Transverse beam dynamics 1

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- Rercommended literature:
  http://cds.cern.ch/record/235242/

Outline

- Beam coordinate system
- Transverse dynamics
  - Motion of particles in beam coordinate system in different **magnetic fields**
  - Strong and weak focusing (!)
  - Mathematical treatment
    - Hill equation
    - The beta-function
    - Matrix optics
    - Beam transport

Very important conclusion, definition

Question to audience
(I require an answer before proceeding)

» Forward reference to something defined later. No problem if you don't understand it right away, you will understand for the 2nd time

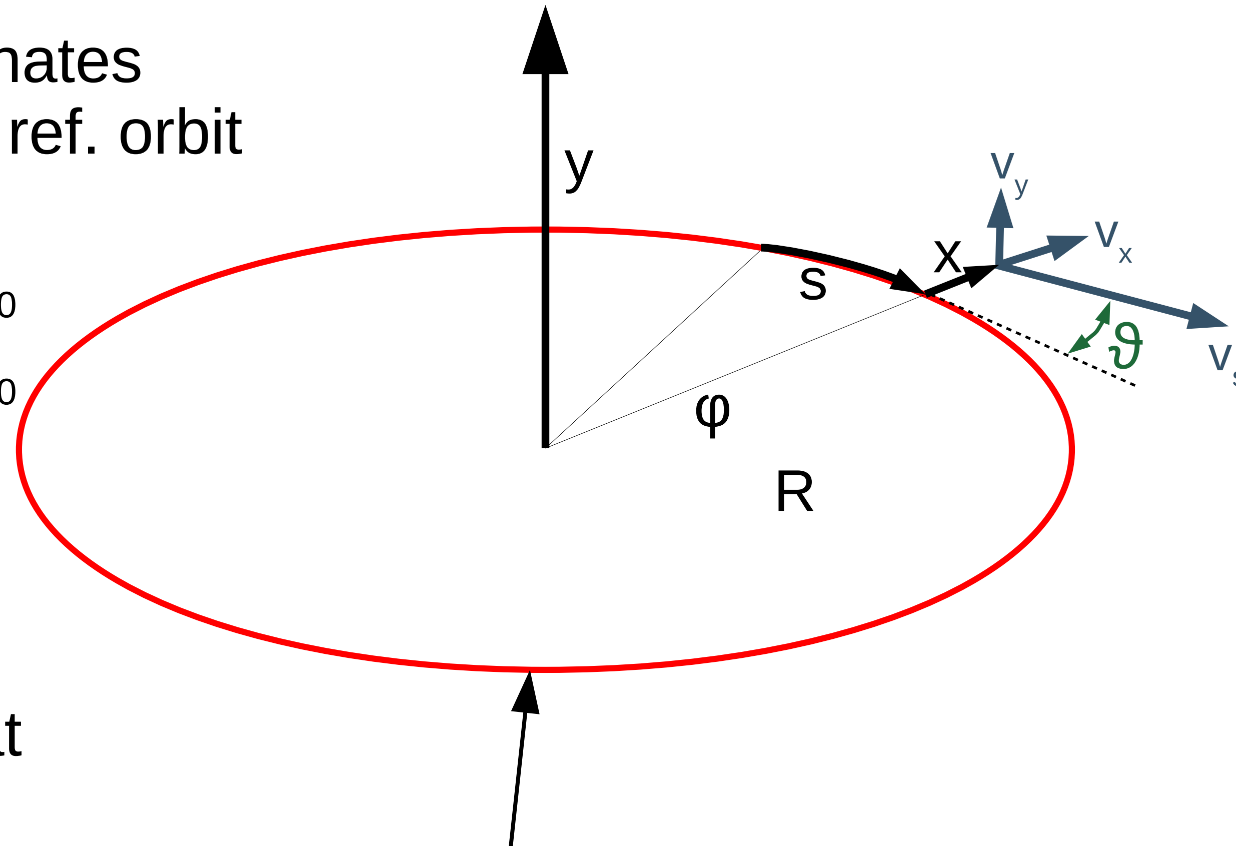
Recommended homework (might be useful for the exam)

Ask it after the lesson, if you are interested

Beam coordinate system

Beam coordinate system

- Position
  - x, y – transverse coordinates
  - s – path length along ref. orbit
- Orbit tangent ≈ angle (ϑ)
  - $x' = dx/ds = v_x / v_s = p_x/p_0$
  - $y' = dy/ds = v_y / v_s = p_y/p_0$



Reference or design orbit
The unique closed orbit of the reference particle with nominal momentum

Phase space variables:

- Transverse: (x,x') (y,y')
  Today we will assume that the two planes decouple, motion in two planes independent

- Longitudinal: (dt, dE)
  Time- and energy-deviation from reference particle

Phase space?

- (x,x') and (y,y') are not really phase-space variables
- Real phase space variables (conjugated coords): (x,p_x), (y,p_y)
- $x' = p_x/p_0$ - proportional to p_x as long as p_0 is constant (i.e. no acceleration) → we can use x’ as a phase-space coordinate
- During acceleration, Liouville’s theorem about the constant area of phase space will not hold in the transverse planes separately for (x,x’) and (y,y’)
- It still holds of course for the true 6-dimensional phase space

Transverse dynamics

Transvese dynamics

- Magnetic force increases proportionally to velocity

$$F = qE + \underbrace{q \cdot v \times B}_{v \approx c\text{, dominates}}$$

- Example: relativistic particle (v=c), B = 1 T = 1 V·s/m^2 (1 Tesla is easy to make)

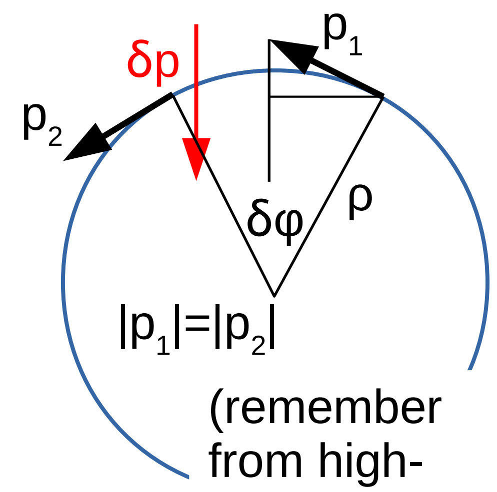
  F = q · 3·10^8 m/s · 1 Vs/m^2 = q · 300 MV/m

  Electric field producing same force as 1 Tesla
  State-of-the-art record: 800 MV/m

- At high momenta we always use magnets!
- At low energies (E_{kin} <≈ 100 keV) electrostatic beamlines are simpler:
  - No currents, no need for cooling
  - Easy to manufacture
  - Magnets can not be controlled precisely at low fields

The effect of magnetic field on the trajectory of charged particles

Magnetic rigidity



(remember from high-school?)

$$F_r = q\cdot v\cdot B = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 2\,p\sin\left(\delta\phi/2\right)/\mathrm{d}t \approx$$

$$\approx p\cdot\delta\phi/\mathrm{d}t = p\frac{v}{\rho} \quad \text{(relativistically true)}$$

radius of curvature

Magnetic rigidity

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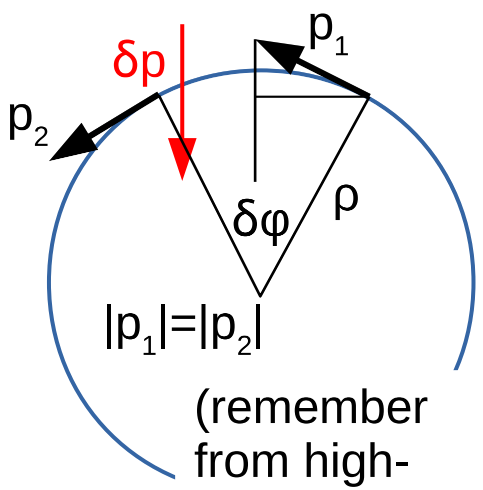


$$B\rho = p/q \quad \text{magnetic rigidity}$$

- In a given magnetic field the particle trajectory depends on p/q only
- B needs to be increased proportionally to p to keep same orbit

Effect of magnetic field

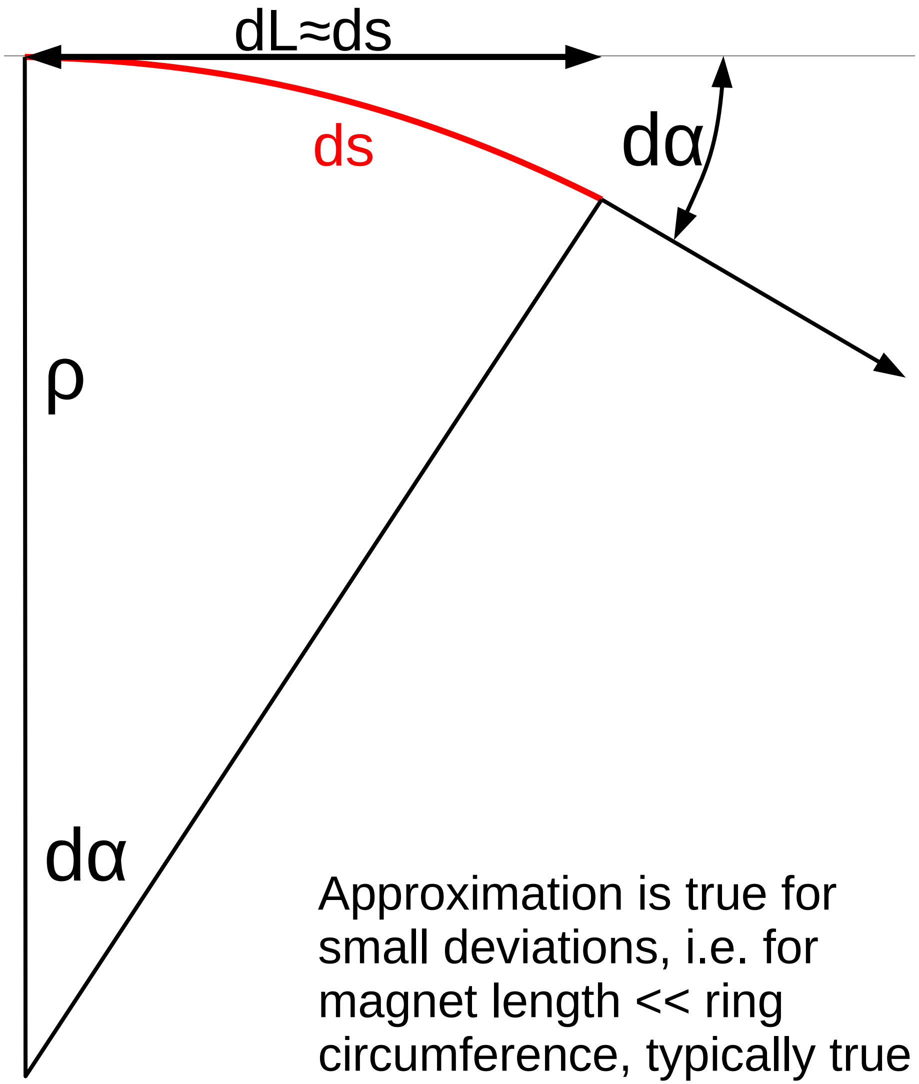
Circular orbit....

$$\mathrm{d}\alpha = \frac{\mathrm{d}s}{\rho} = \frac{B\cdot \mathrm{d}s}{B\cdot\rho} = \frac{B\cdot \mathrm{d}s}{p/q}$$

$$\alpha = \frac{1}{p/q}\int B\cdot \mathrm{d}s \approx \frac{1}{p/q}\int B\cdot \mathrm{d}L$$

Important property of a given magnet: total **bending power** = integrated field

Normalisation by magnetic rigidity



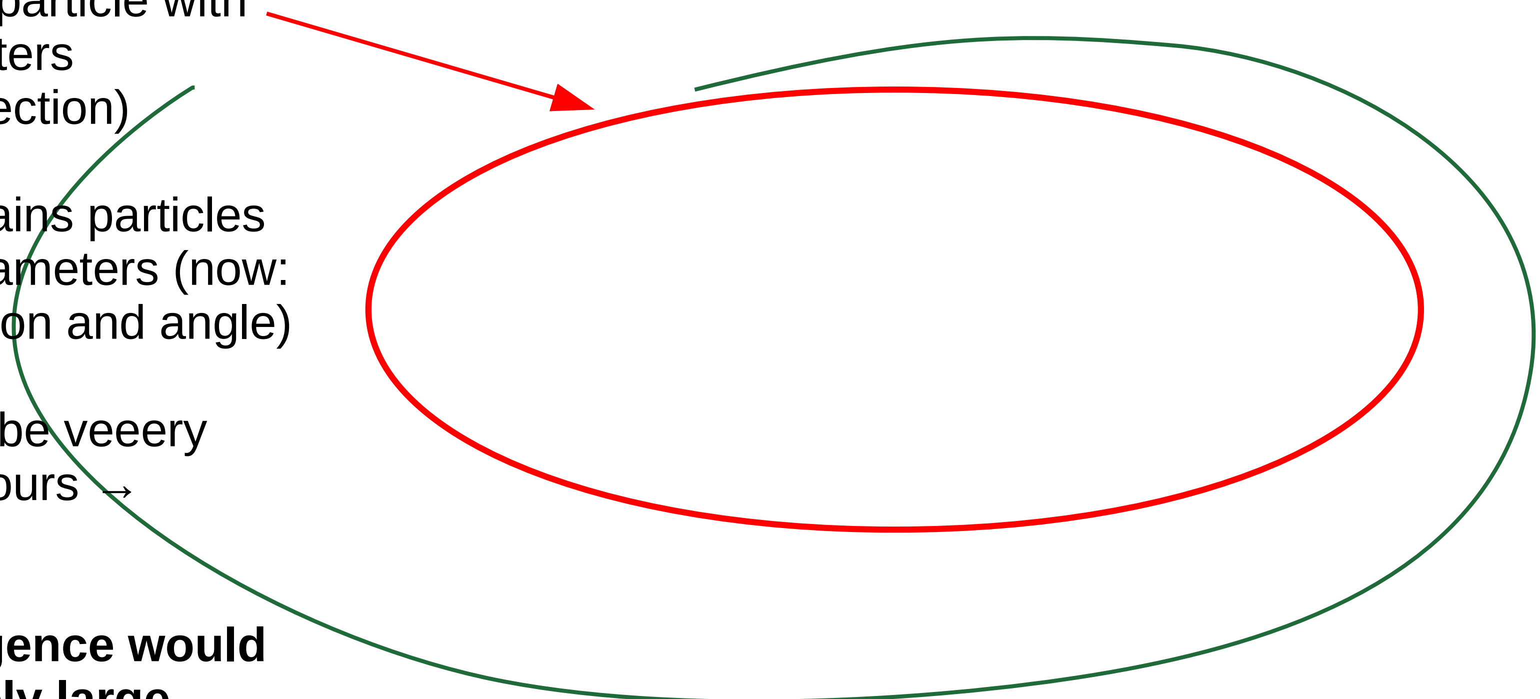
Approximation is true for small deviations, i.e. for magnet length << ring circumference, typically true

The nominal/design orbit

- Reference orbit is the trajectory of the particle with nominal parameters (momentum, direction)

Realistic, oscillating orbits

- Reference orbit is the trajectory of the particle with nominal parameters (momentum, direction)
- Real beam contains particles with varying parameters (now: transverse position and angle)
- Path length can be veeery long (LHC: 10 hours → s=10^{10} km !!)
- **Smallest divergence would lead to extremely large beamsize. Need focusing!**



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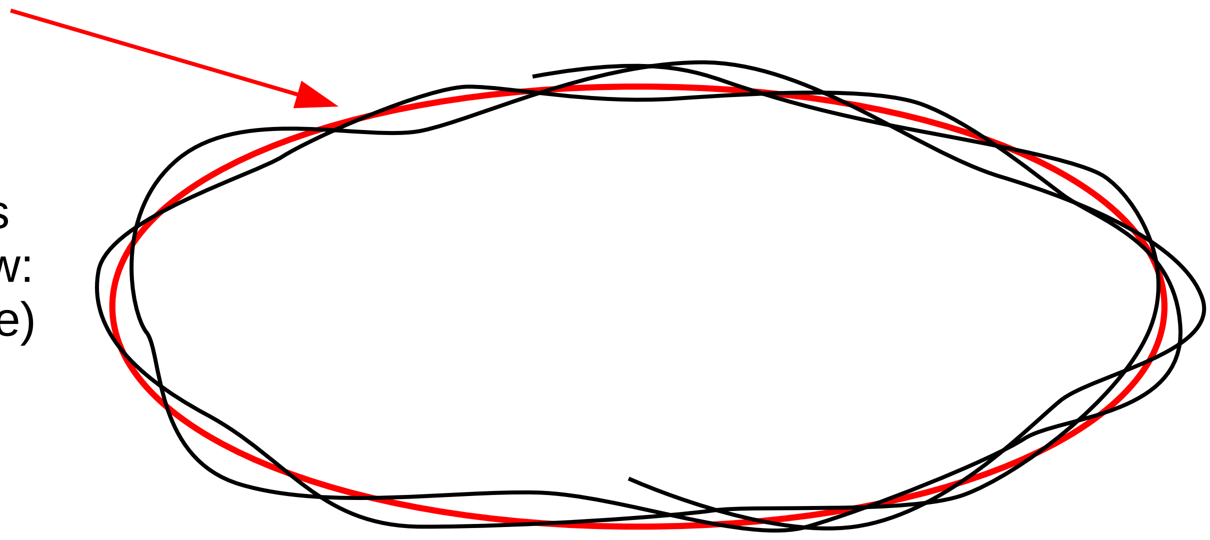
- Stability: small perturbations of the initial conditions should not cause large deviations at later times. Need bounded solutions → **focusing**!

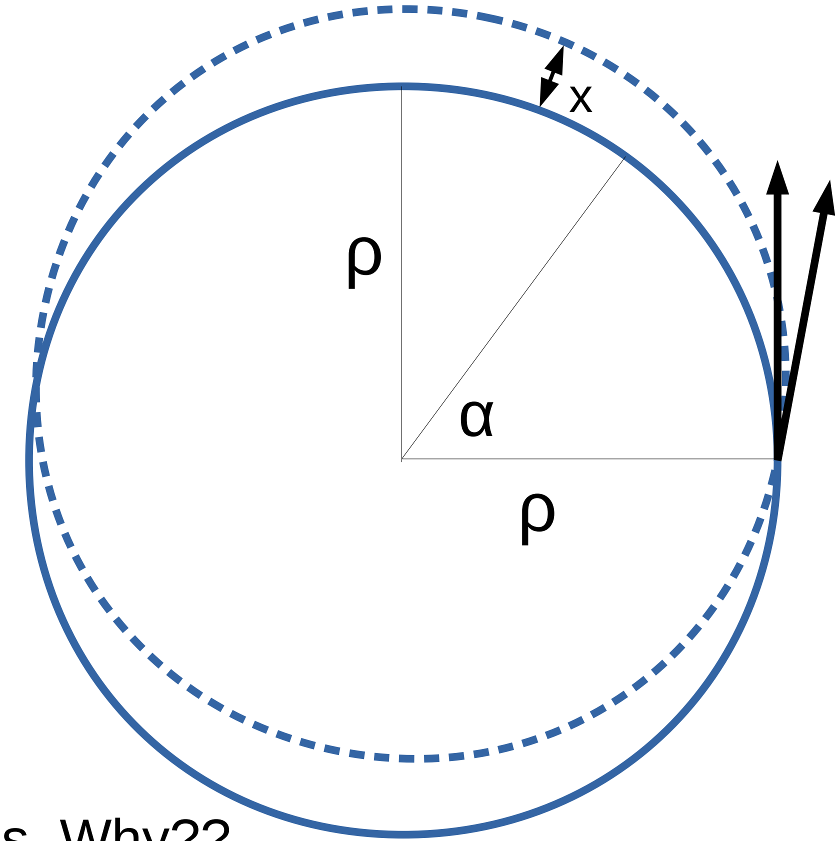


Betatron oscillation:
Oscillation of the particle orbits around the reference orbit

Weak focusing

The key to weak focusing: geometric focusing

- In a homogeneous B field: all orbits are identical circles
- Particle starting from same spot but with “wrong” angle returns to the nominal orbit after 1 complete turn
- Oscillation with a period of the circumference: 2ρπ around the reference orbit



$$x \approx \sin(\alpha) = \sin(s/\rho)$$

$$x'' = -\frac{x}{\rho^2}$$

- That is: a homogeneous B field focuses. Why??

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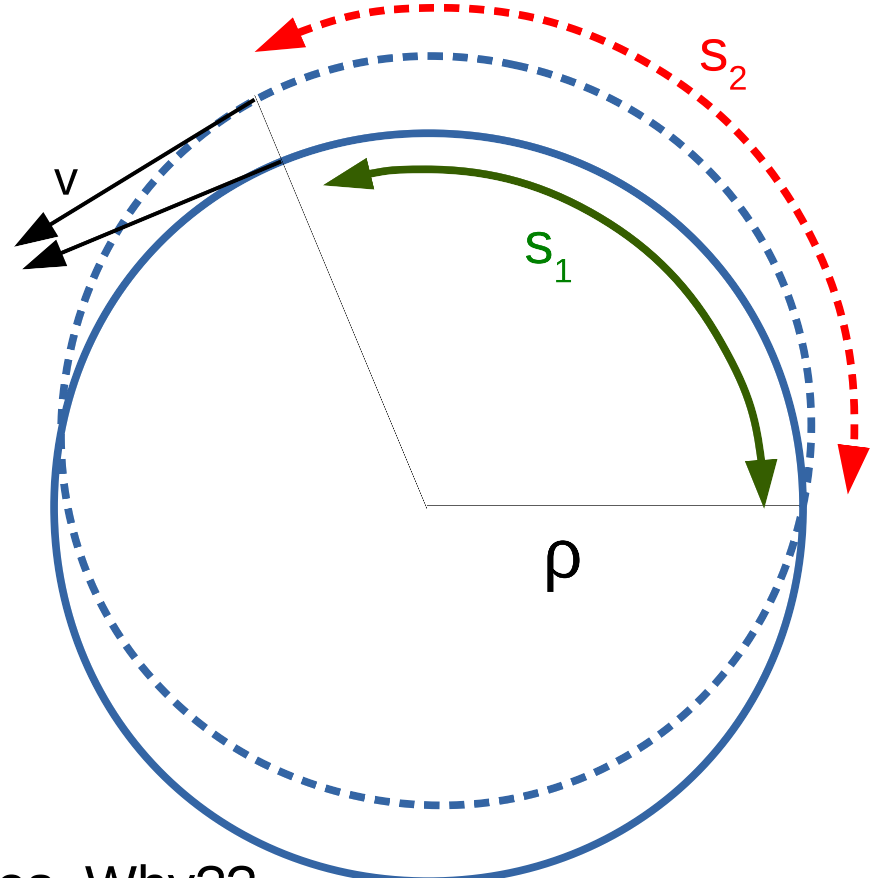

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- That is: a homogeneous B field focuses. Why??
- **External orbit has a longer path ($s_2 > s_1$), experiences the bending force of the magnetic field for a longer time → larger deviation**

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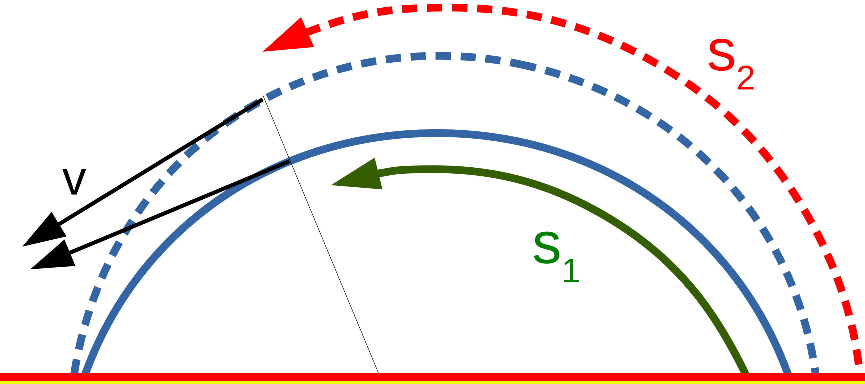

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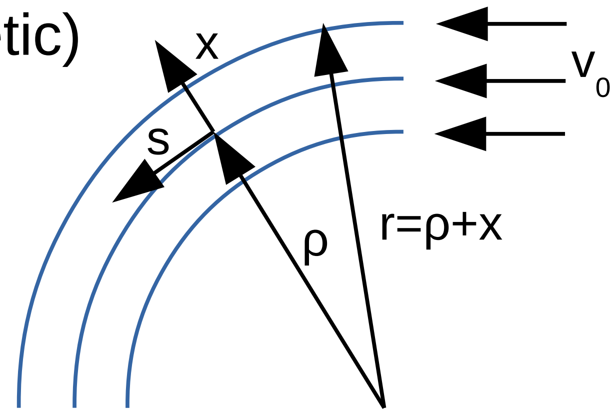
- **External orbit has a longer**
  force of the magnetic field

Geometric focusing:
- In a bending magnetic field, with a **curved reference orbit**
- Due to the fact that more external orbits have a longer path in the field
- and therefore experience a larger deviation
- For dipole magnets: $x'' = - x/\rho^2$

Weak focusing: bending plane

- Naively: B decreasing with R will not focus (smaller force acting on particles starting already more outside)
- Due to geometric focusing we can still allow a slight decrease
- Limiting case of focusing: parallel (monoenergetic) beam should remain parallel



$$B r = \frac{p}{q}$$ magnetic rigidity

$$B_{\text{limit}}(r) = \frac{p}{q}\frac{1}{r} = \frac{p}{q\rho}\left(\frac{\rho}{r}\right) \equiv B_0 \cdot \left(\frac{r}{\rho}\right)^{-1}$$ ρ = nominal beam radius

- For B ~ 1/r the orbits are concentric circles. Parallel particles remain parallel. Limiting case of focusing
- If B decreases with R faster than this → defocusing in the bending plane

Weak focusing: bending plane

- Definition of field index (gradient of B(r) on reference orbit)

$$n=-\frac{\rho}{B_y(\rho)}\frac{\mathrm{d}B_y}{\mathrm{d}r}\bigg|_{r=\rho} \qquad \text{corresponds locally to: } B_y(r)=B_y(\rho)\left(\frac{r}{\rho}\right)^{-n}$$

Make dimensionless by this factor

- B_{limit} decreases as 1/r → n_{limit}=1.
- **Horizontal focusing if $B_y(r)$ decreases slower than $B_{limit}(r)$ → n<1**
- without geometric focusing we would need n<0 (i.e. increasing field) for horizontal focusing
- If $B_y(r)$ decreases faster than $B_{limit}(r)$, i.e. n>1 → no focusing in the bending plane

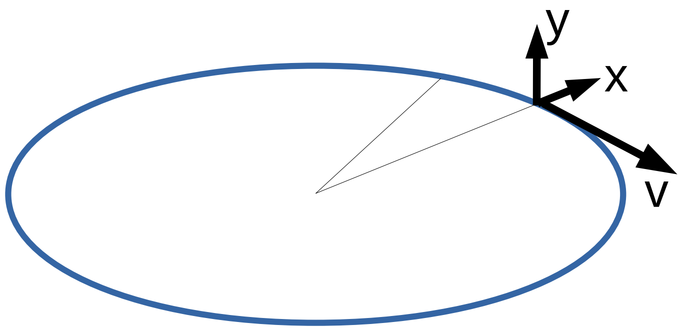
Weak focusing: non-bending plane

- To focus in non-bending plane: need B_x
- φ-invariance: B_φ (or B_s) =0
- Maxwell-equation

$$\nabla \times B = 0$$

$$(\nabla \times B)_\phi = \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial r} = 0 \quad \text{(cylindrical coordinate system)}$$

$$\left.\frac{\partial B_x}{\partial y}\right|_{r=\rho} = \left.\frac{\partial B_y}{\partial r}\right|_{r=\rho} = -\frac{B_y(\rho)}{\rho} n$$



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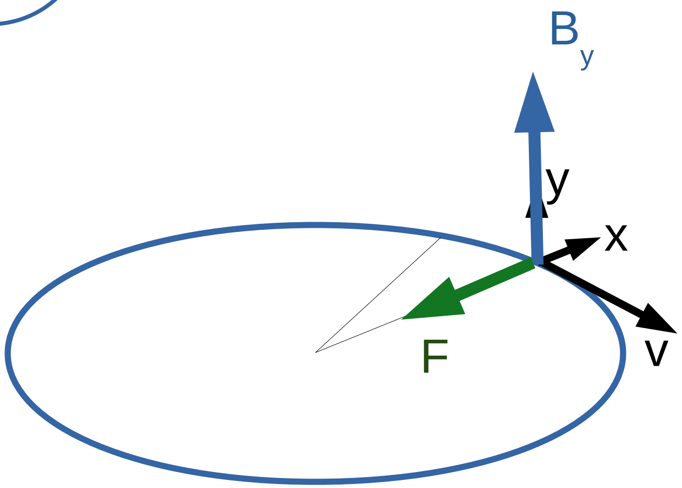
- For positive q:
  - $B_y(\rho)>0$ to keep on orbit, positive



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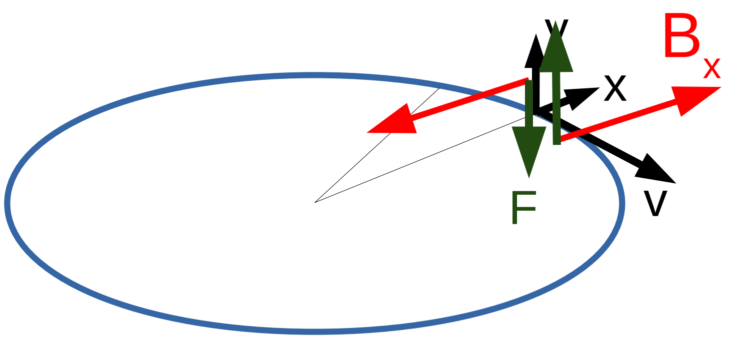
  - For vertical focusing: $\partial B_x/\partial y < 0$, negative



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For vertical focusing: need **n>0**

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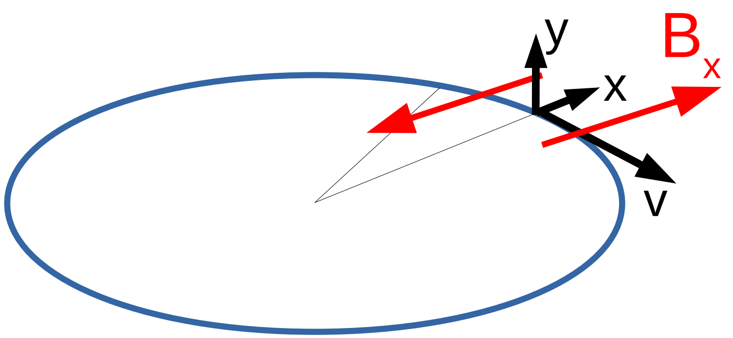

Weak focusing = simultaneous focusing in x/y

- Horizontal focusing if n<1 (the smaller n, the stronger focusing)
- Vertical focusing if n>0 (the larger n, the stronger focusing)
- **0< n < 1** – simultaneous focusing in both planes
- 0< n → B_y decreases with r
- Constant B field (n=0) was weakly focusing in bending plane (1 oscillation in 1 turn)
- 0<n **focuses even weaker!** Less than 1 oscillation within 1 turn. Large excursion from reference orbit

For weak focusing the tune (Q) is smaller than 1

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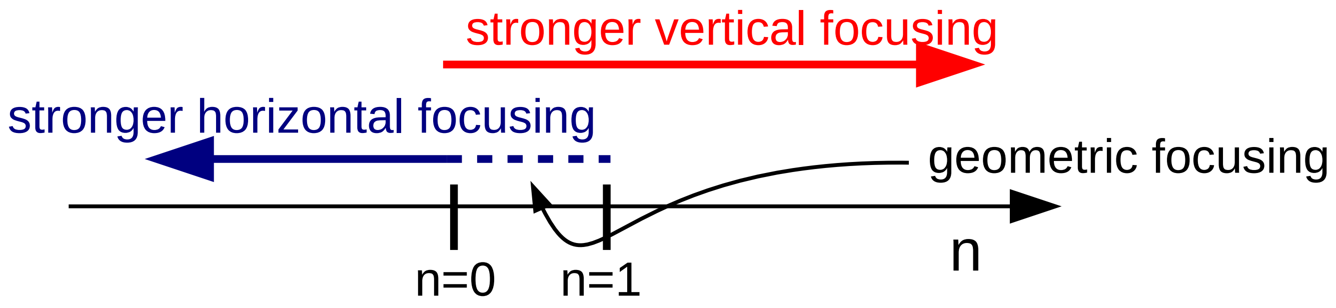
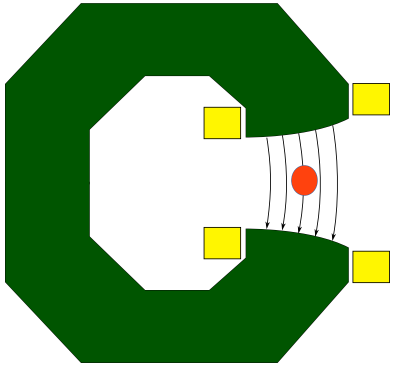


- Bad compromise, can not make focusing stronger in one plane without loosing it in the other plane



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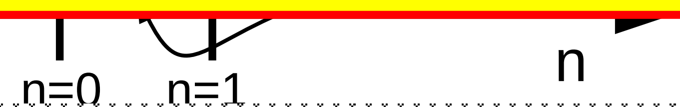
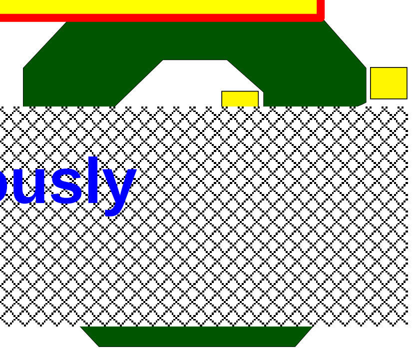
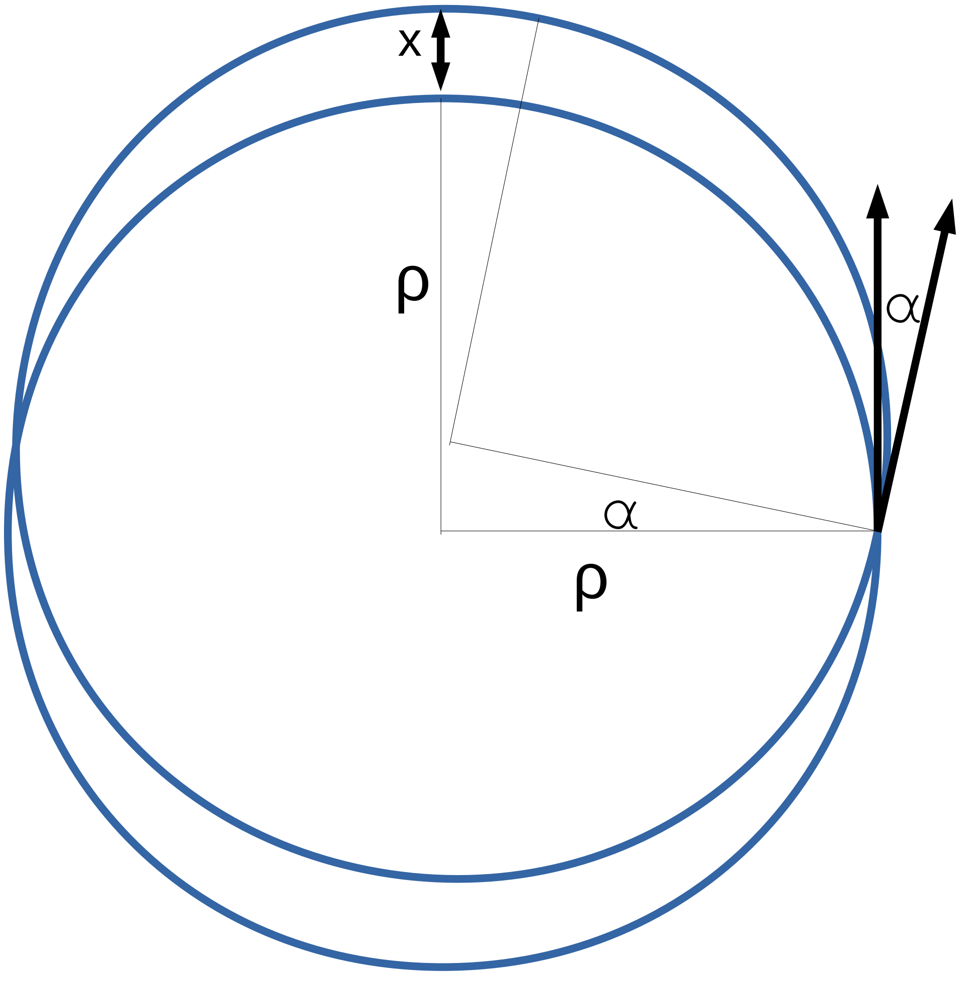
Weak focusing is only possible for curved reference orbits because geometric focusing only works here





How can we focus in both planes simultaneously for straight reference orbits???

Weak focusing: bending plane



Problem of weak focusing: large oscillations

For the illustrated case, B=const:

$x \approx \rho \cdot \alpha$

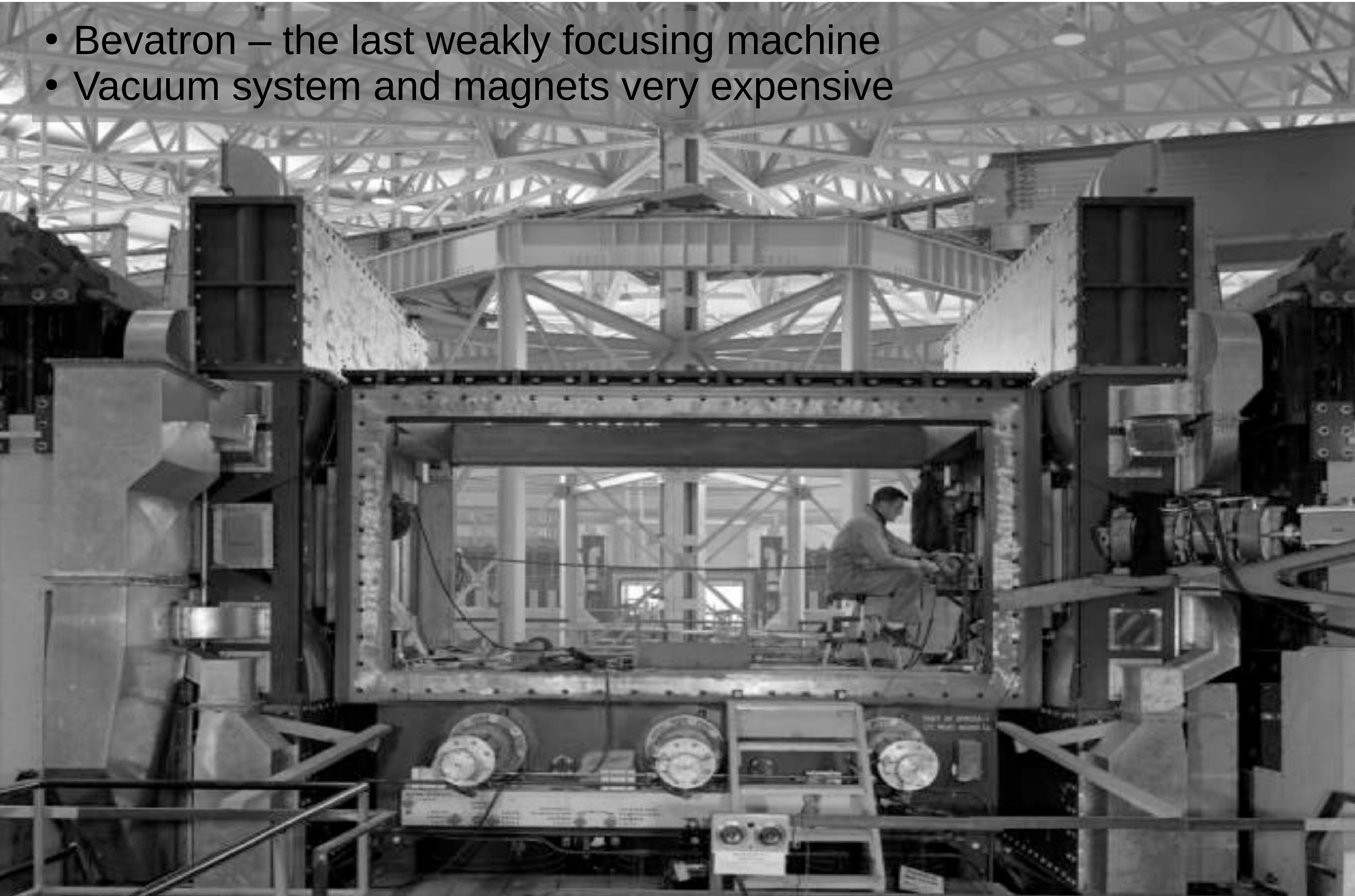
Typical divergence: α=1 mrad
Typical ring radius: ρ=100 m
Oscillation amplitude
$x \approx 10$ cm

(For weak focusing the tune (Q) is even smaller, slower oscillations, even larger beamsize)

Bevatron's beampipe size: 1.2 m

Drawback of weak focusing: large beamsize

- Bevatron – the last weakly focusing machine
- Vacuum system and magnets very expensive

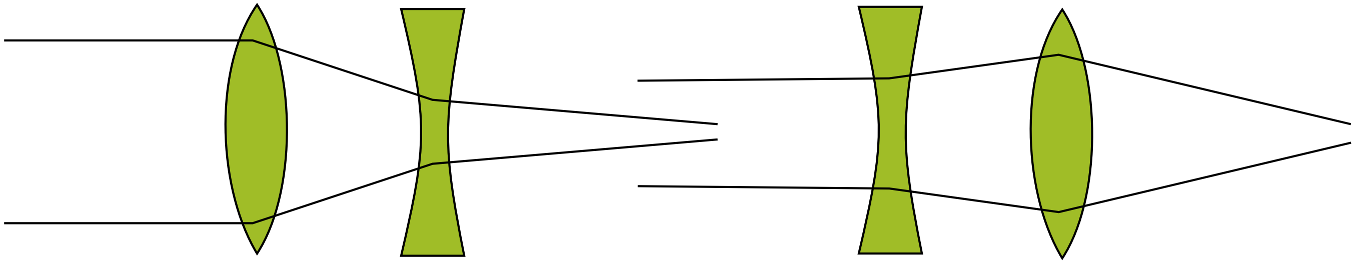


Strong focusing

Strong focusing

- Do not need simultaneous focusing in both planes
- No limits, can use large field gradients
- Alternating focus-defocus gives a stronger net focusing than for weak focusing
- Strength and distance of the lenses must of course be adapted

Optical analogy:



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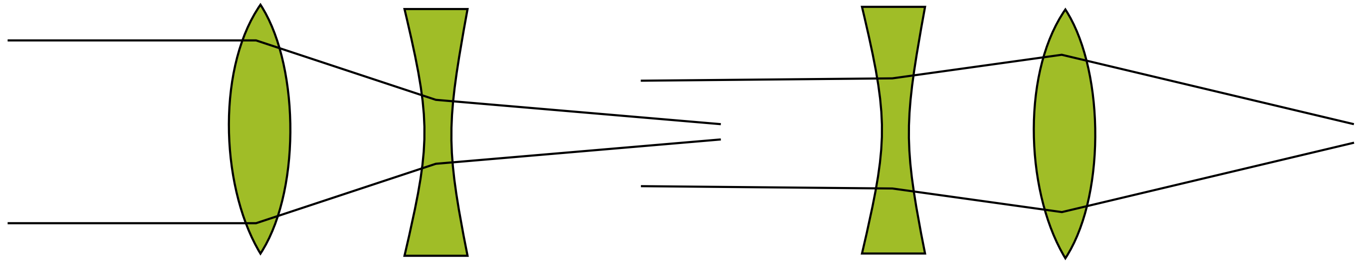

What is the ponderomotive force, how is it related to strong focusing?

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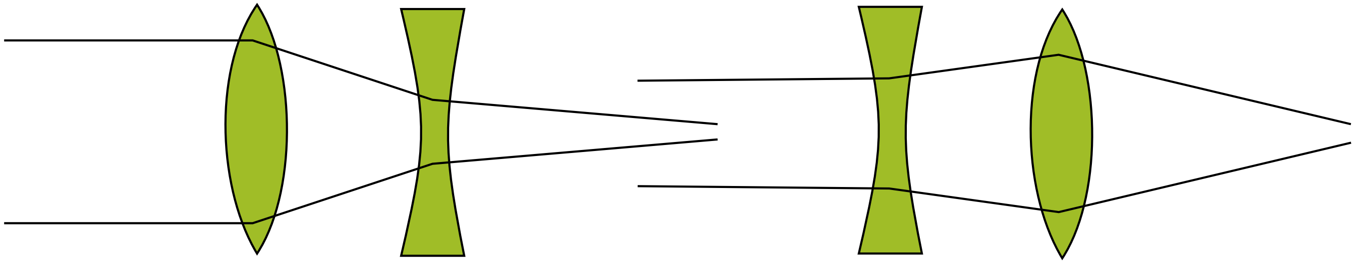
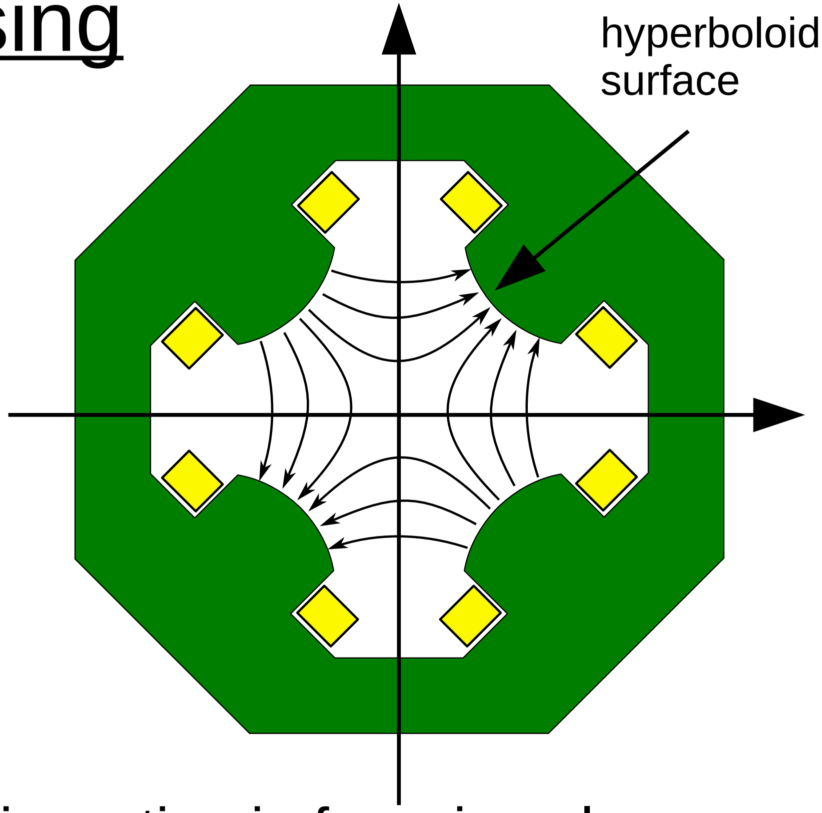


What is common in strong focusing and RFQ? (…. what is an RFQ, btw???)

Strong focusing

- Standard device: quadrupole magnet (next term in the Taylor series after the dipole term)
  $B_x = g\,y$; $B_y = g\,x$
  (g = field gradient – depends on magnet current)

  $F = q\,v \times B$
  $F_x = -q\,v\,B_y = -q\,v\,g\,x$
  $F_y = \;\; q\,v\,B_x = +q\,v\,g\,y$
  Force proportional to position → harmonic motion in focusing plane
  Defocusing (exponential runaway) in the other plane



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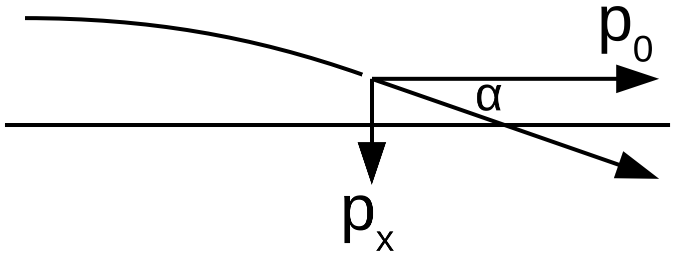
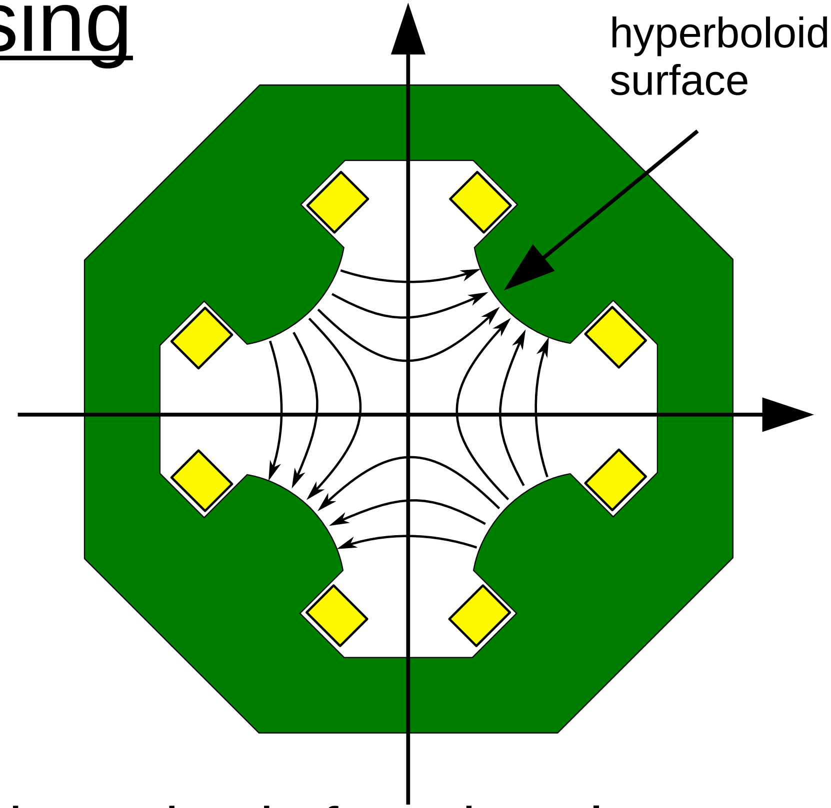


$$\frac{\mathrm{d}\,x'}{\mathrm{d}\,s} = x'' = -\frac{g}{p/q}\,x \equiv -k\,x \quad \text{(focusing plane)}$$

$$\frac{\mathrm{d}\,y'}{\mathrm{d}\,s} = y'' = \frac{g}{p/q}\,y \equiv k\,y \quad \text{(defocusing plane)}$$

Strong focusing



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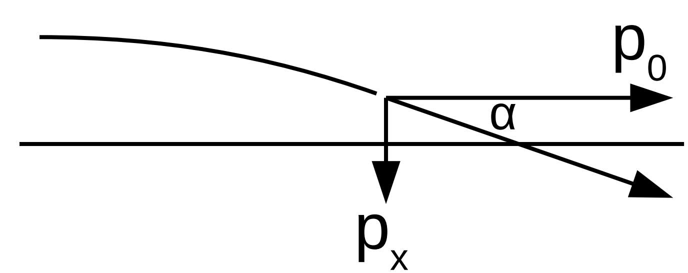

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Field gradient [T/m] normalized by magnetic rigidity

Strong focusing – combined function magnets

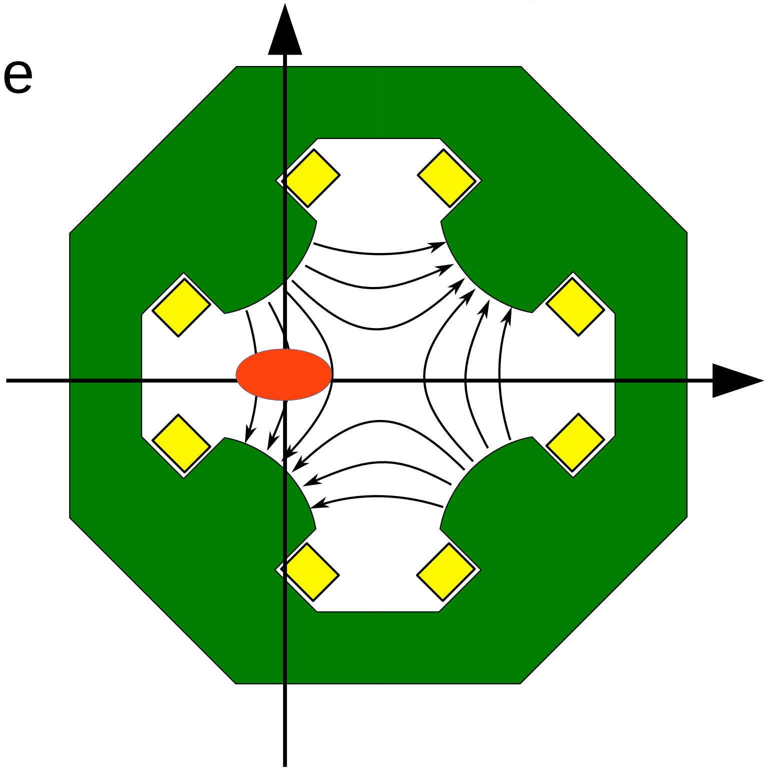
- Move the magnet in the horizontal plane

$B_x = -g\,y$

$B_y = B_0 - g\,x$ dipole+quadrupole

deflection, keeping on orbit

focus/defocus



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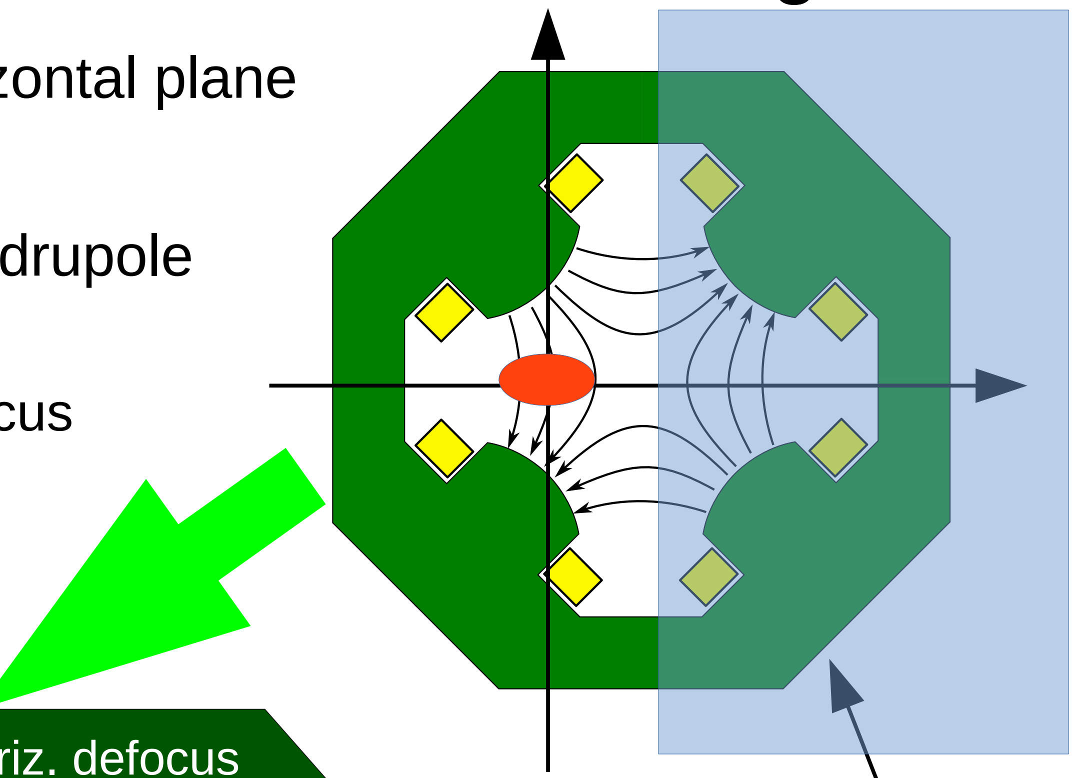

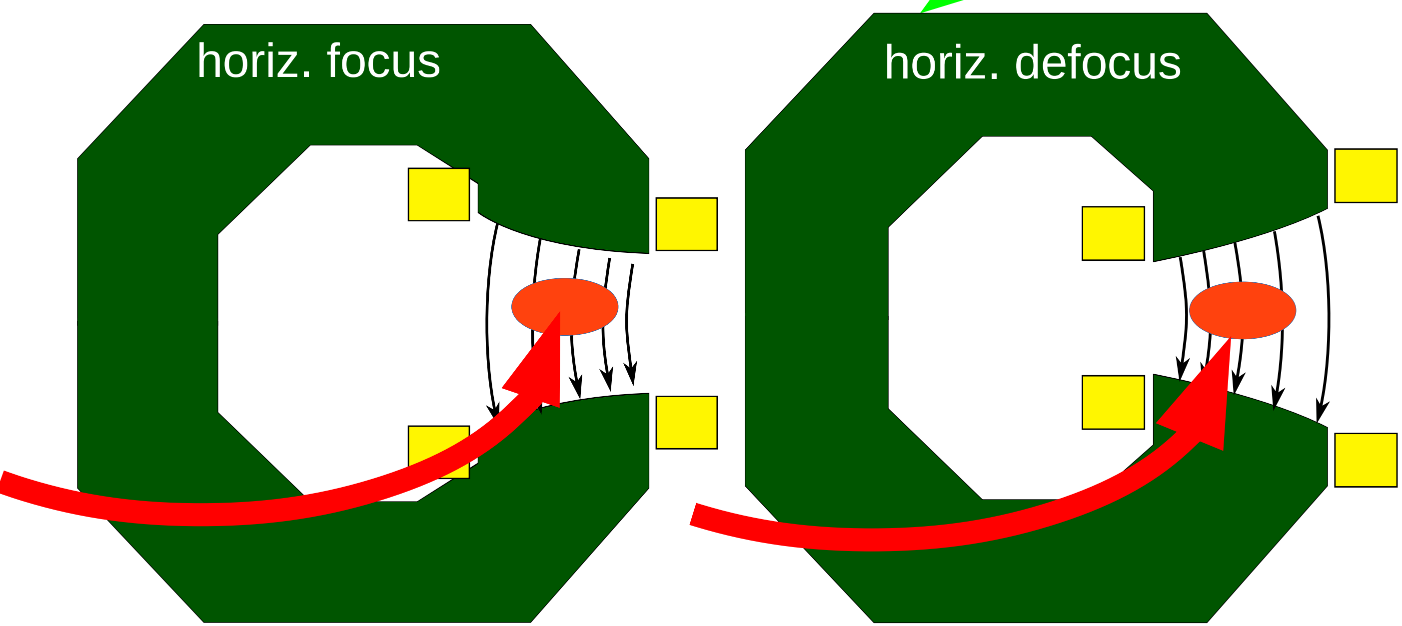

- Same magnet used for bending and focusing
- AGS and CERN PS

Strong focusing – combined function magnets

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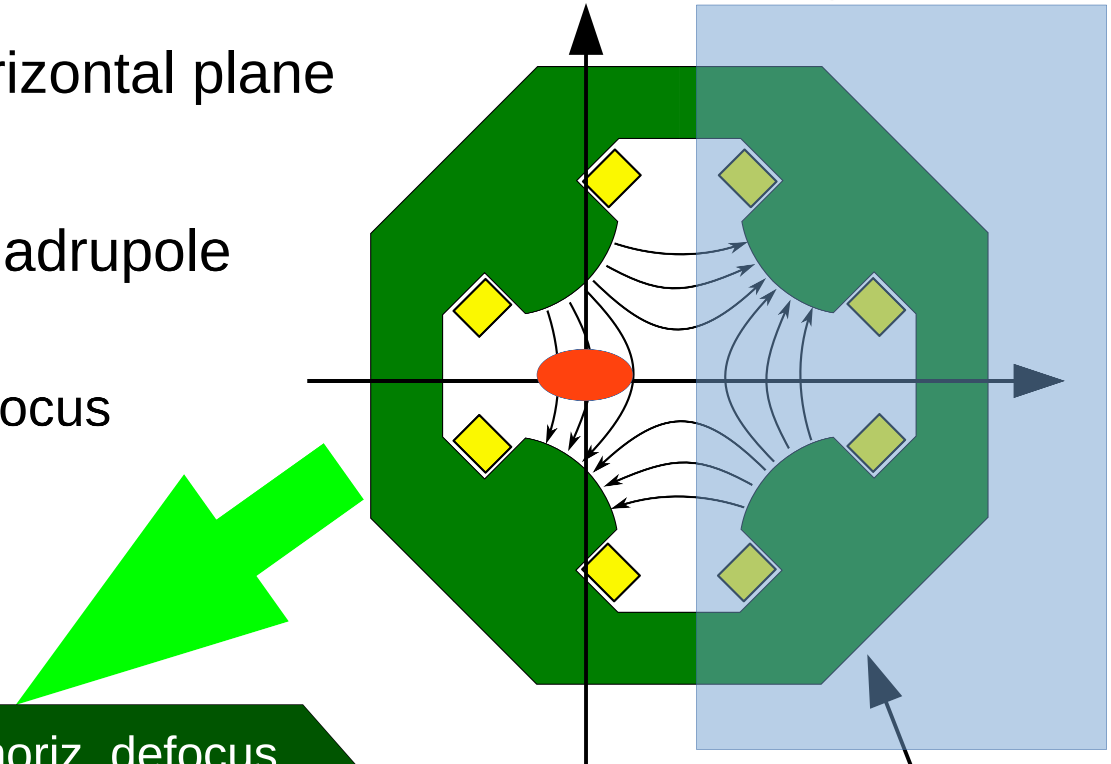
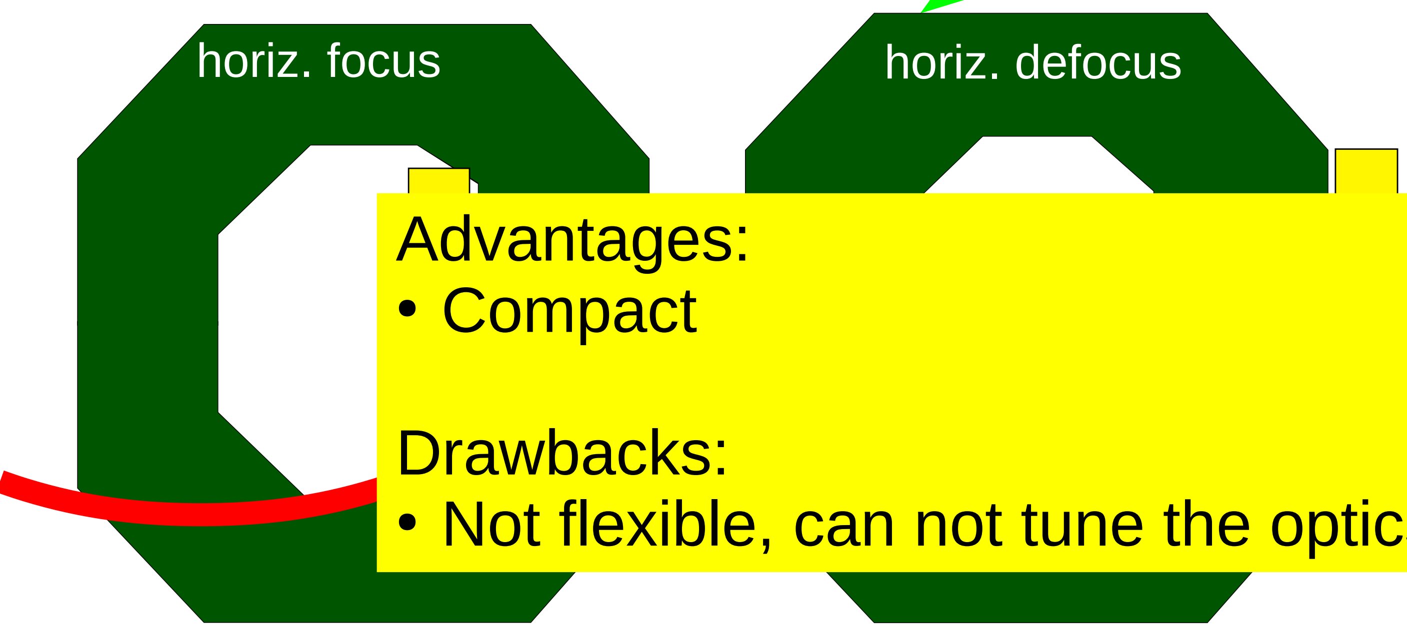
dipole+quadrupole

focus/defocus

deflection, keeping on orbit



This half is not used



Advantages:
- Compact

Drawbacks:
- Not flexible, can not tune the optics

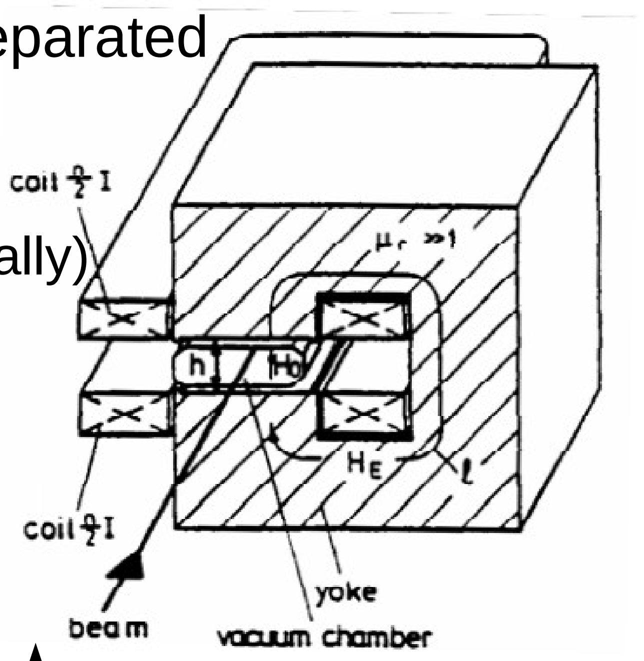
net used for

d focusing

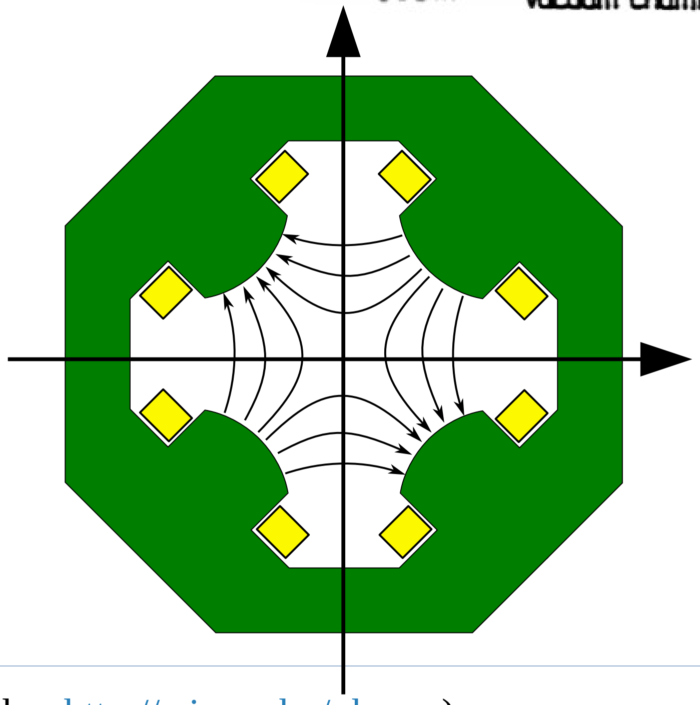
- AGS and CERN PS

Strong focusing – separate function magnets

- In modern synchrotrons the two functions are separated
  - Bending magnets: dipoles
    trajectories are curved
    reference trajectory enters perpendicularly (typically)
    Only geometric focusing



  - Focusing magnets: quadrupoles
    centered on the beam
    reference orbit is straight



  - Correctors:
    - dipoles for orbit correction
    - higher-order (sextupole, octupole) magnets for other corrections

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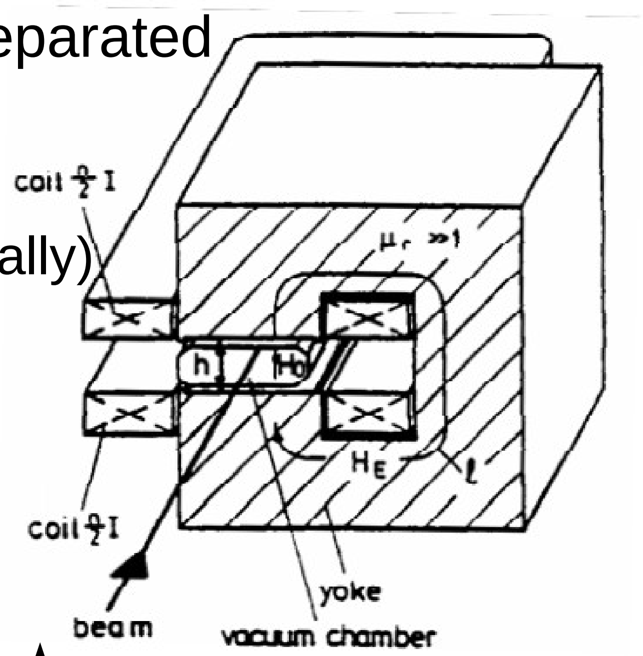

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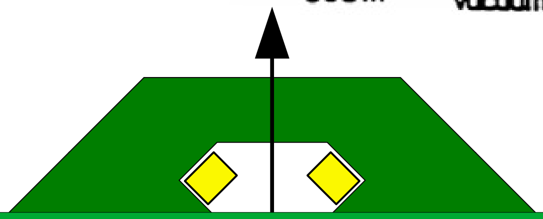
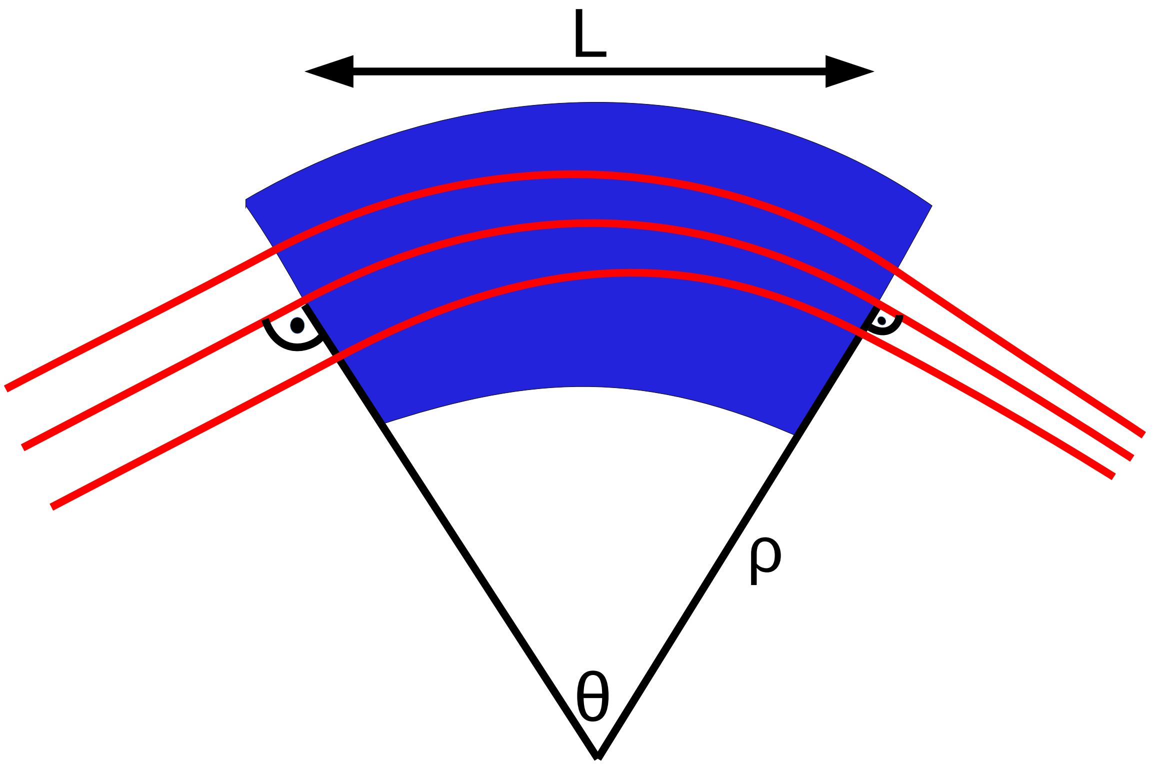


What is the difference between a strong-focusing and a weak-focusing magnet?

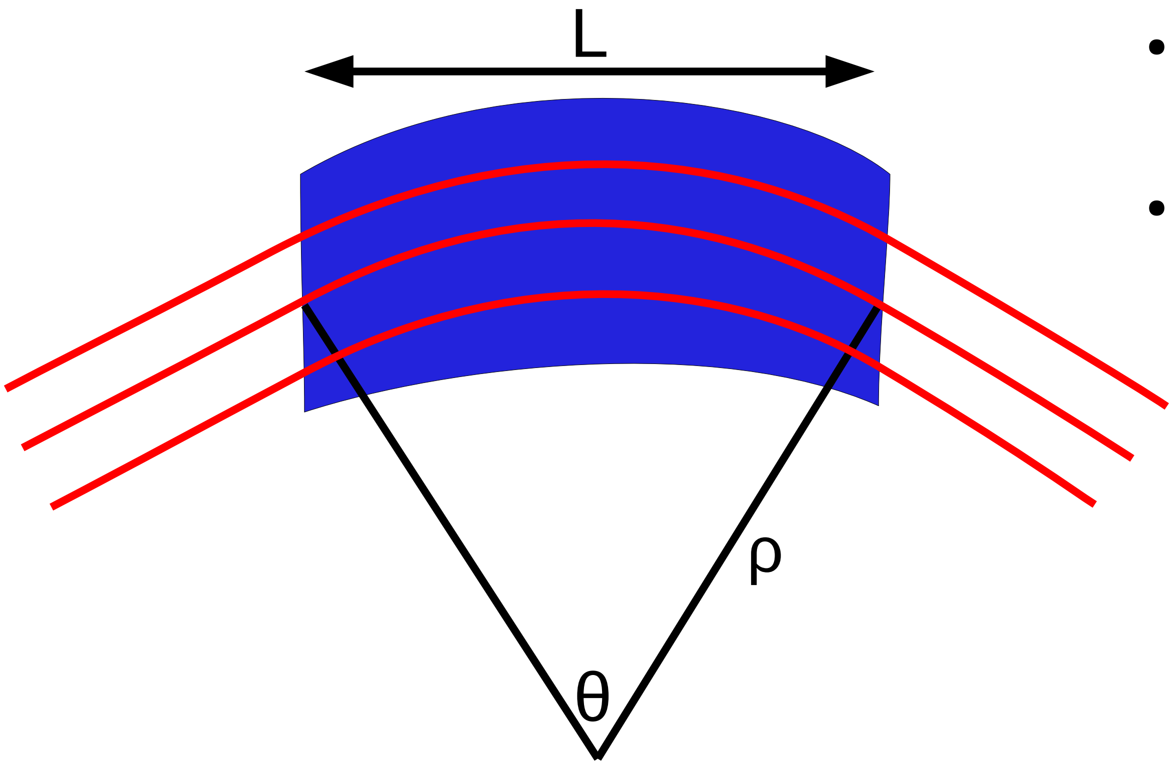
What is the difference between a combined-function magnet and a separate-function magnet?

Bending magnets: sector magnet



- Reference orbit enters and exits perpendicularly
- **No vertical focusing** (earlier: edge focusing needs non-perpendicular entry)
- **Horizontal** plane: **geometric focusing**

Bending magnets: parallel face magnets



- Parallel beam remains parallel. **No horizontal focusing.**
- Reference orbit enters and exits the magnet at an angle

Bending magnets: parallel face magnets

- Parallel beam remains parallel. **No horizontal focusing.**
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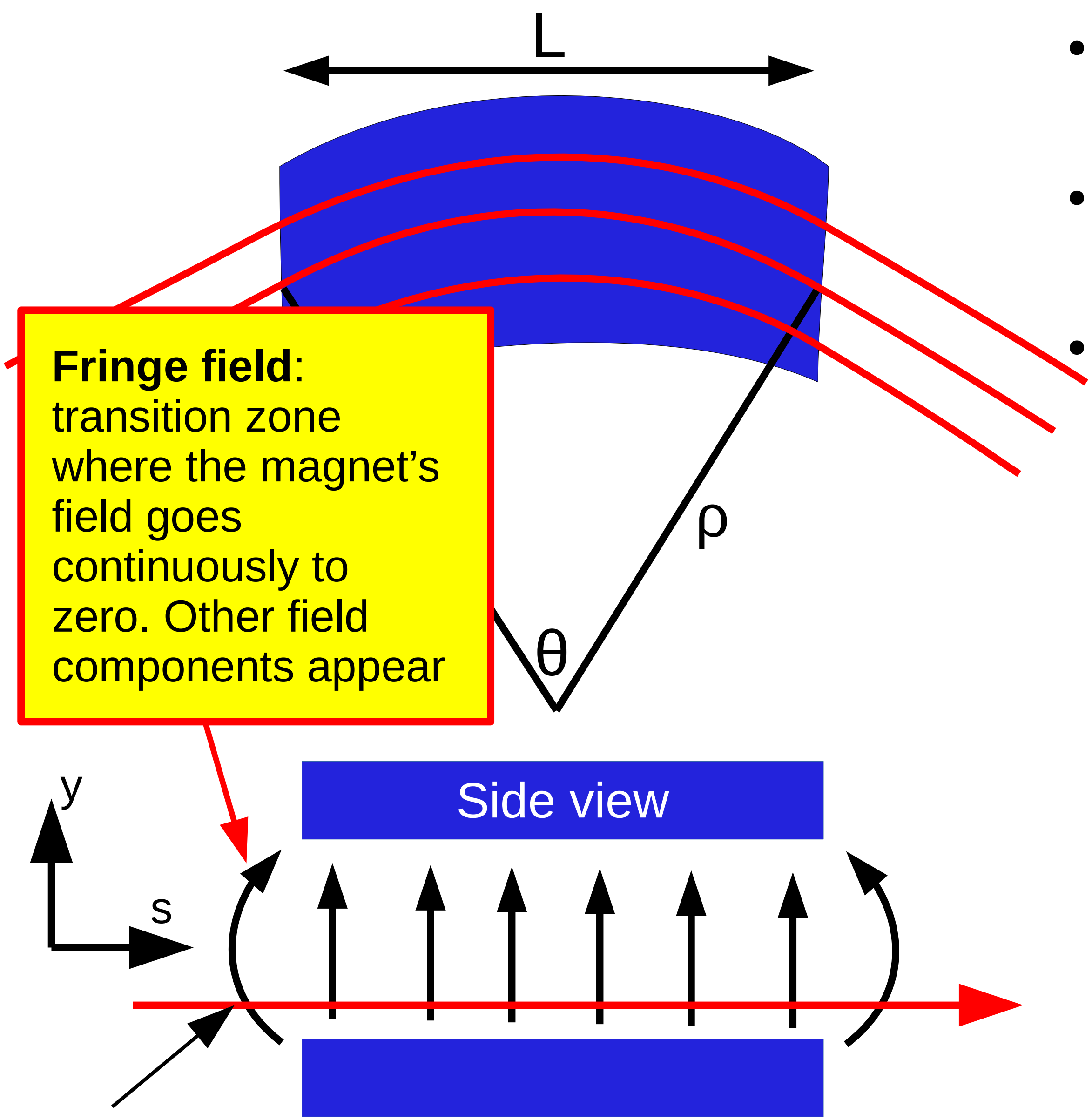
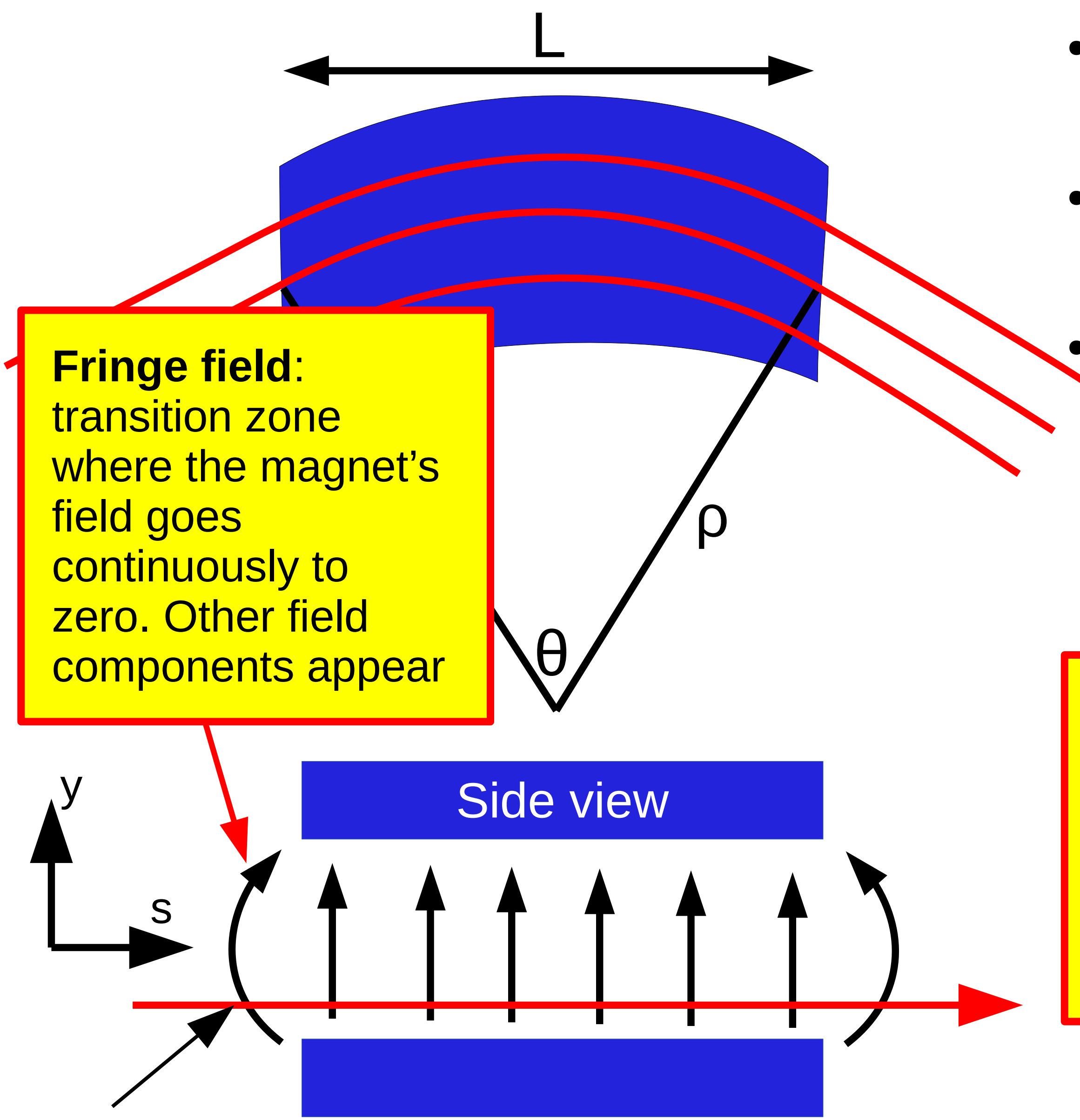
- This gives **vertical focusing** (edge focusing):

  v_x x B_s ~ F points to the central plane



v_x is perpendicular to the plane of this plot

Bending magnets: parallel face magnets



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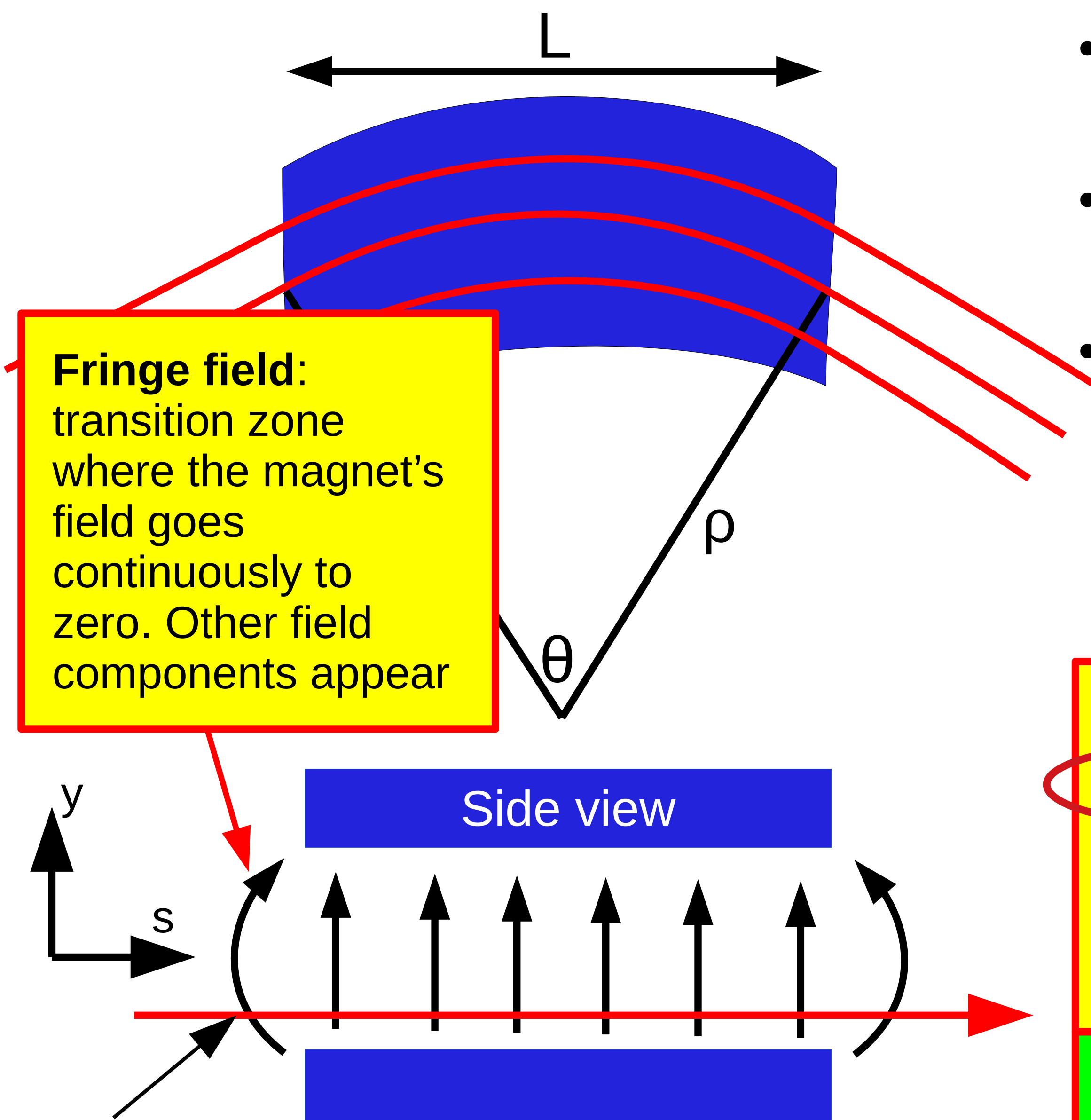
Edge focusing: vertical (de)focusing in the fringe field of the magnet due to particles entering the magnet non-perpendicularly

Bending magnets: parallel face magnets



Fringe field: transition zone where the magnet's field goes continuously to zero. Other field components appear

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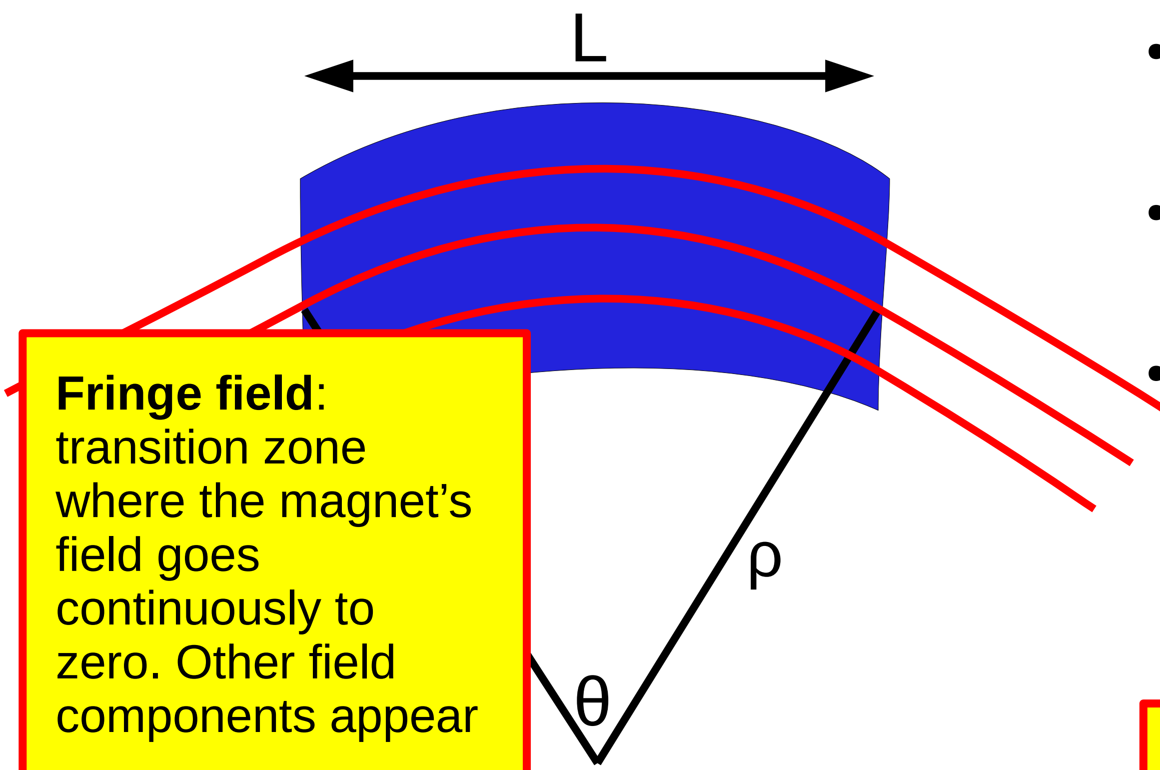
Edge focusing: vertical (de)focusing in the fringe field of the magnet due to particles entering the magnet non-perpendicularly

Is this particular magnet in the figure defocusing or focusing vertically?

Bending magnets: parallel face magnets



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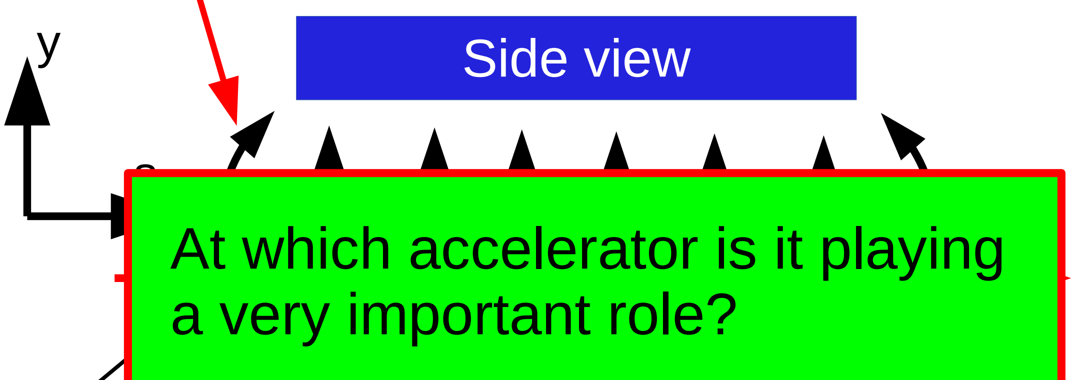


At which accelerator is it playing a very important role?

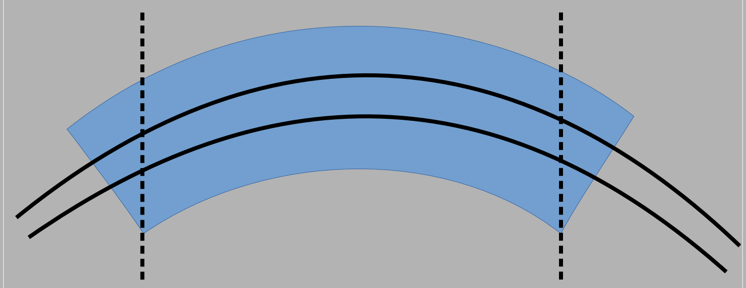
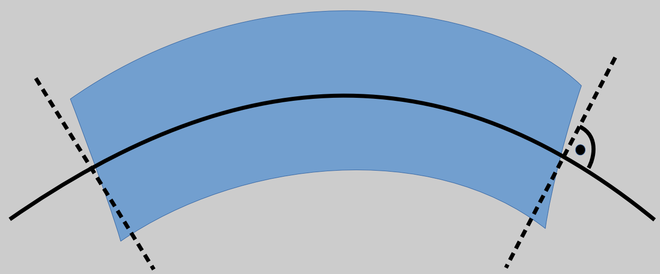
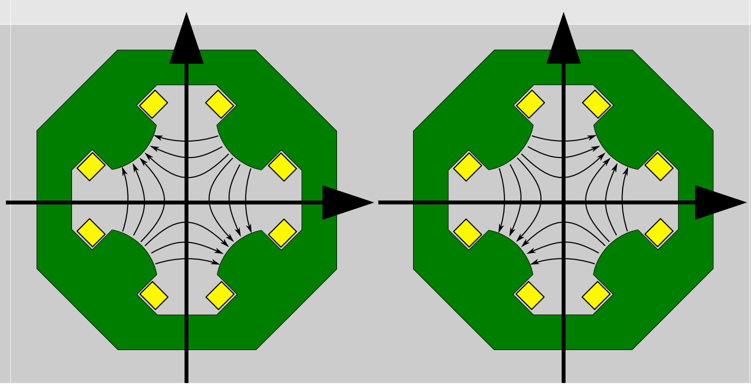
v_x i … is plot

Edge focusing: vertical (de)focusing in the fringe field of the magnet due to particles entering the magnet non-perpendicularly

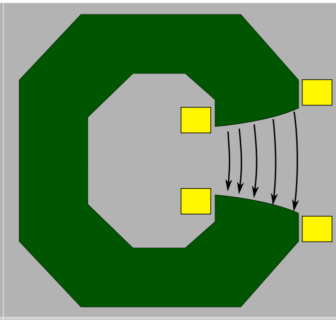
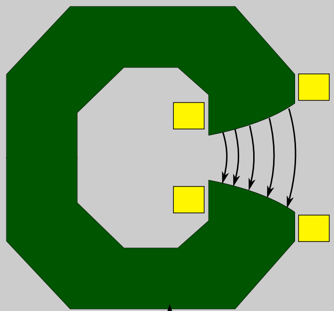
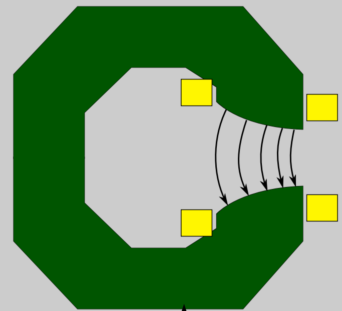
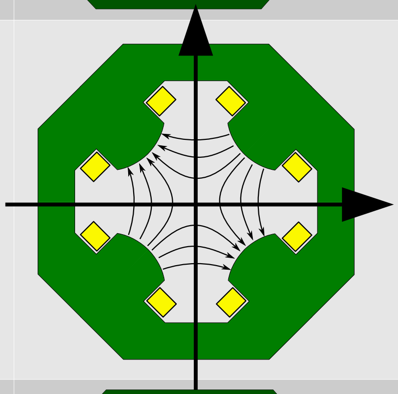
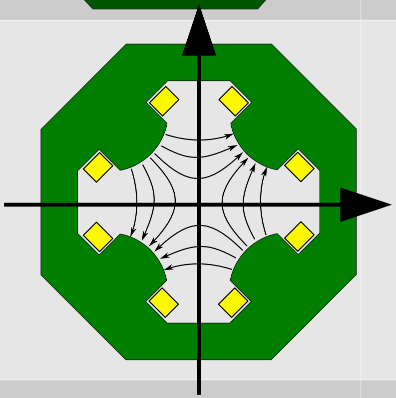
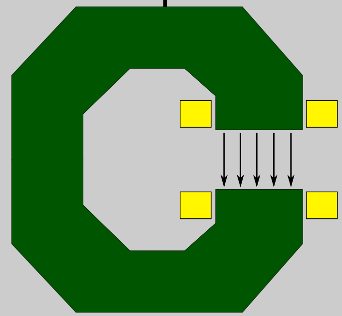
Dipole magnets

- If deviation is small
  - sector dipole ≈ parallel face dipole
  - focusing is negligible

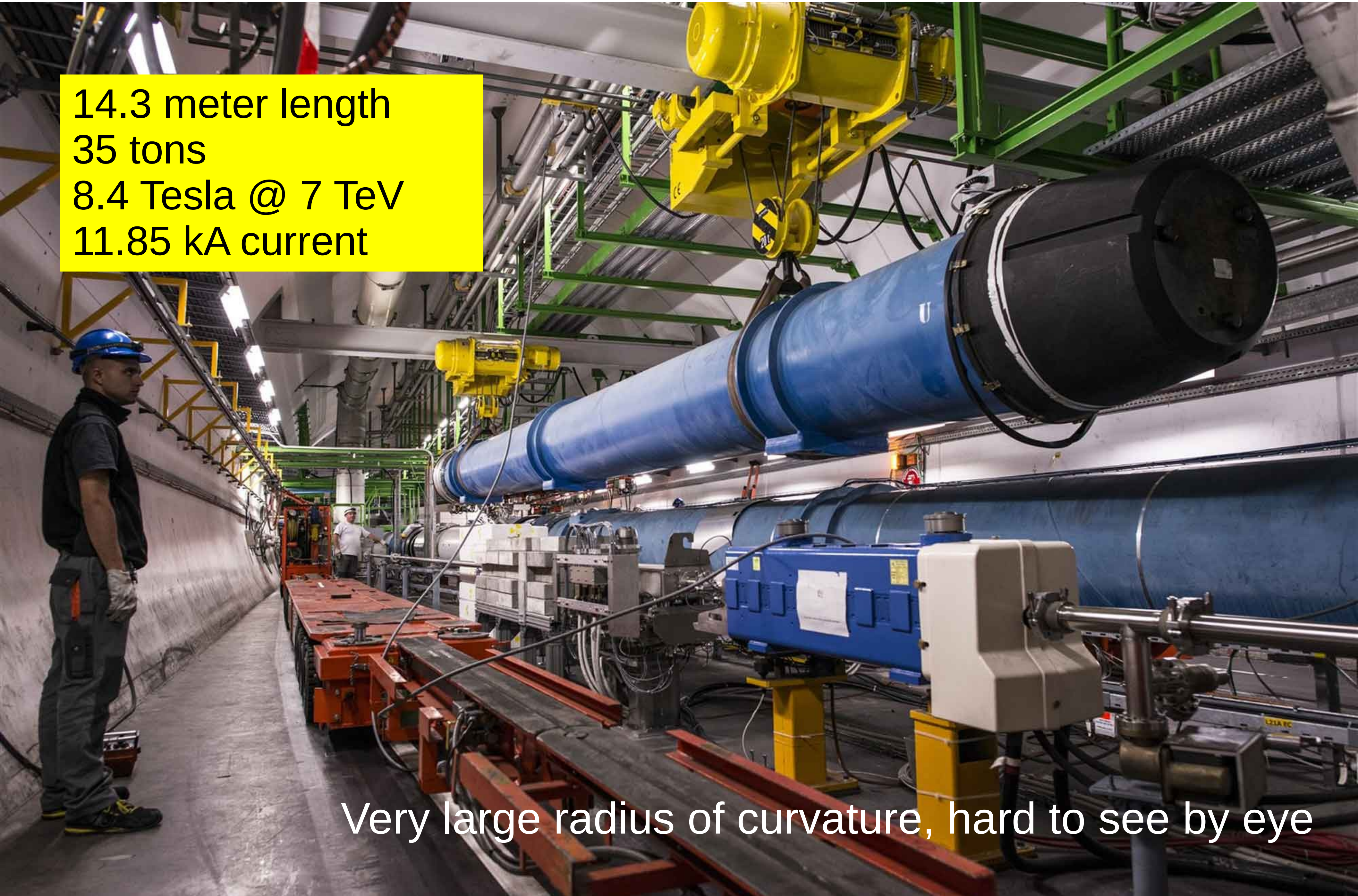
Focus lexikon

Geometric focusing		For curved reference orbit, reference planes and magnets entry- and exit-planes are non-parallel, particle running more outwards has a longer path in B field, receives larger bending Distributed focusing along the magnet's entire length
Edge focusing		Reference orbit enters/exits non-perpendicularly. Focusing/defocusing in the vertical plane (Illustrated: AVF cyclotron) Localized effect at the magnet ends
Weak focusing	Properly chosen field gradient $0 < n < 1$	Simultaneous focusing in x/y. Possible due to geometric focusing
Strong focusing		Quadrupoles with alternating polarity/gradient. Net effect is a simultaneous focusing in both planes, stronger than weak focusing

Magnet lexikon

Weak focusing magnet			Focuses and bends. Non-alternating, weak gradient, decreases with R Necessarily combined function
Strong focusing magnet	Combined function magnet	 	Focuses and bends Alternating, strong gradient + dipole
	Separate quadrupole	 	Focuses only Alternating, strong gradient, no dipole
Bending magnet			Dipole field (possible focusing effects: geometric, edge – depending on face orientation)

LHC dipole magnet

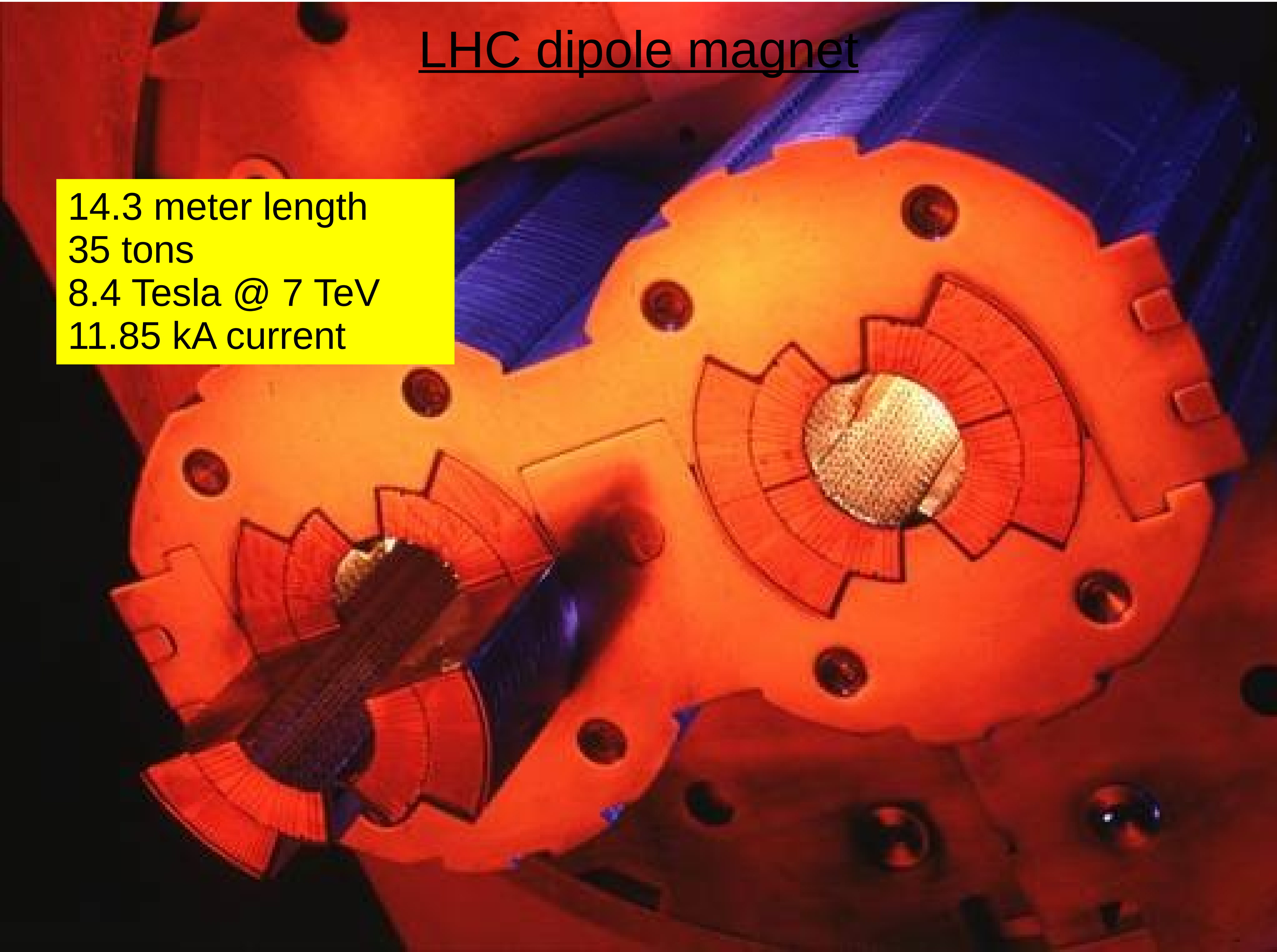


14.3 meter length
35 tons
8.4 Tesla @ 7 TeV
11.85 kA current

Very large radius of curvature, hard to see by eye

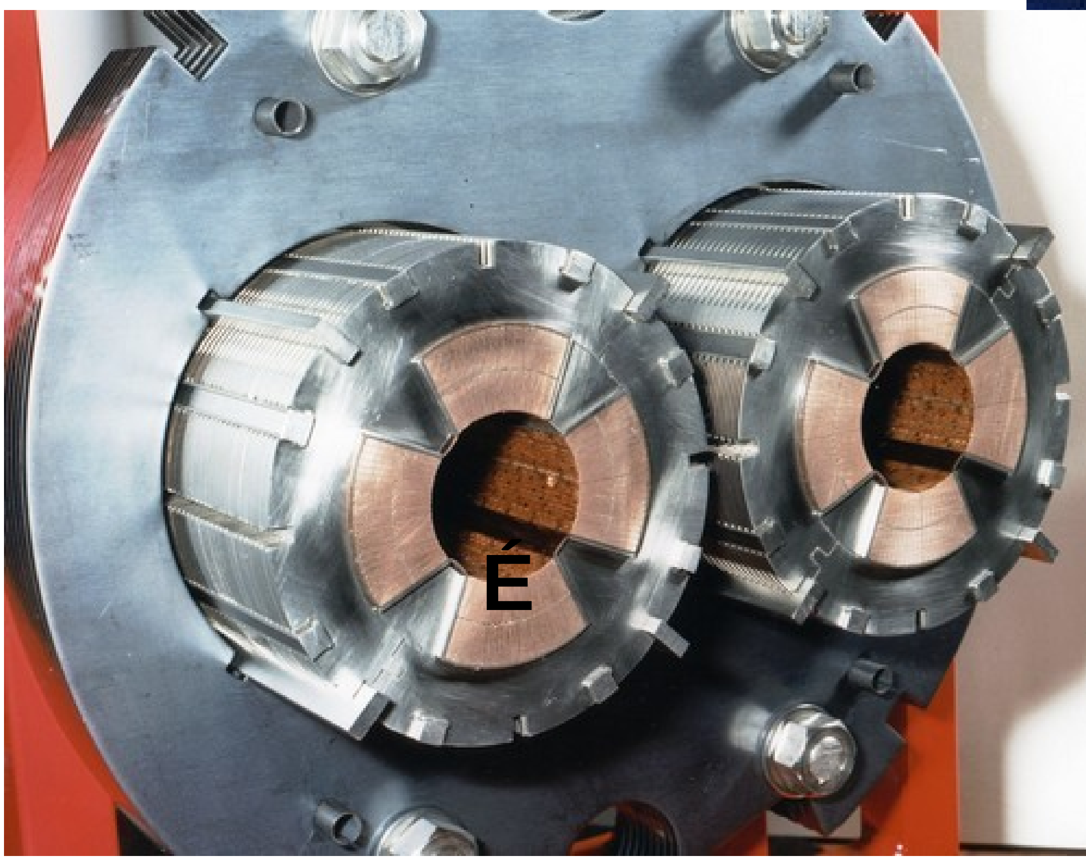
LHC dipole magnet

14.3 meter length
35 tons
8.4 Tesla @ 7 TeV
11.85 kA current



Quadrupole magnets

Normal-conducting magnet with iron yoke



LHC superconducting dipole

Mathematical treatment

Transverse dynamics: Hill's equation

- Assumption: motion in two planes independent
- We have seen for a transverse coordinate x the following equation holds in general

$$x'' + k(s)\, x = 0$$

periodic (ring)

Hill equation

Second-order linear differential equation with a periodic coefficient

Similar to the harmonic oscillation, but "spring strength" is changing with "time" (s)

- $k(s) = 0$ in drift space (no magnetic field)
- $k(s) = \pm g/(p/q)$ - quadrupole strength
- $k(s) = 1/\rho^2$ in the bending dipole, in the bending plane (ρ is nominal orbit)
- k(s) = quadrupole + dipole: $g/(p/q) + 1/\rho^2$ in combined function magnets
- ... smoothly changing in between
- (approximately)

Solution of Hill's equation

$x''+k(s)x=0$ like a harmonic equation.

Look for the solution in similar form:

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\cos(\phi(s)+\phi_0)$$

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Parameterization has ambiguity

E.g. ε → 2·ε, β → β/2 gives same function

Solution of Hill's equation

$x'' + k(s)x = 0$ like a harmonic equation

Look for the solution in similar form:

$$x(s) = \sqrt{\beta(s)}\sqrt{\epsilon}\cos(\phi(s) + \phi_0)$$

Substitute into Hill's equation:

$$\cos(\phi + \phi_0)\left\{\frac{\beta''}{2\sqrt{\beta}} - \frac{\beta'^2}{4\beta^{3/2}} - \sqrt{\beta}\phi'^2 + k\sqrt{\beta}\right\} + \sin(\phi + \phi_0)\left\{\frac{-\beta'\phi'}{\sqrt{\beta}} - \sqrt{\beta}\phi''\right\}$$

$$\beta = \frac{1}{\phi'}$$ with this choice: 0

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Substitute here and require that it is zero

$$\beta=\frac{1}{\phi'}$$ with this choice: 0

$$\beta\cdot\beta''-\frac{\beta'^2}{2}+2k(s)\cdot\beta^2=2$$

- Second order **nonlinear** diff. equation
- Has multiple solutions. To choose one, we need to fix for example, pl: β(0), β'(0)
- How should we choose?

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- Second order **nonlinear** diff. equation
- Has multiple solutions. To choose one, we need to fix for example, pl: β(0), β'(0)
- How should we choose?

In a ring k(s+L)=k(s) (periodic)

Let's choose the periodic solution:

β(L)=β(0)
β'(L)=β'(0)

(Discuss why it works for a nonlinear equation...)

Solution of Hill's equation

$x''+k(s)x=0$ like a harmonic equation

Look for the solution in similar form:

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\cos(\phi(s)+\phi_0)$$

Substitute into Hill's equation:

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- Second
- Has mu
  fix for ex
- How sh

In a ring k(s+L)=k(s) (periodic)

Let's choose t

β(L)=β(0)
β'(L)=β'(0)

With these prescriptions – **requiring periodicity** - the β(s) function is a **unique** function of the longitudinal position (s) along the ring, and depends only on the ring's optics: k(s)

Solution of Hill's equation

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\,\cos(\phi(s)+\phi_0)$$

- β unique and periodic, with same period as k(s)
- β and φ are not independent: $\phi' = 1/\beta$
- φ has an ambiguity: can be shifted by a constant.
  The **phase advance** is unique and same for all particles:

$$\Delta\phi \equiv \phi(s_2)-\phi(s_1)=\int_{s_1}^{s_2}\frac{\mathrm{d}s}{\beta}$$

- 2nd order diff. eq. → 2 constants of motion:

$\sqrt{\epsilon}$ 'amplitude'

ϕ_0 initial phase

- [β] = m "Beta function"
- [ε] = m·rad = m (in practice: mm mrad)

Twiss parameters

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\,\cos[\phi(s)+\phi_0]$$

$$x'=-\sqrt{\frac{\epsilon}{\beta(s)}}\sin[\phi(s)+\phi_0]-\alpha(s)\sqrt{\frac{\epsilon}{\beta(s)}}\cos[\phi(s)+\phi_0]$$

$$\alpha(s)\equiv-\beta'(s)/2 \qquad \gamma(s)\equiv\frac{1+\alpha^2}{\beta(s)}$$

- The following holds (check it!):

$$\gamma(s)x^2+2\alpha(s)x\,x'+\beta(s)x'^2=\begin{pmatrix}x & x'\end{pmatrix}\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix}=\epsilon$$

The φ_0 constant disappeared.
Contains only ε in a very simple form

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Which curve is described by this equation??

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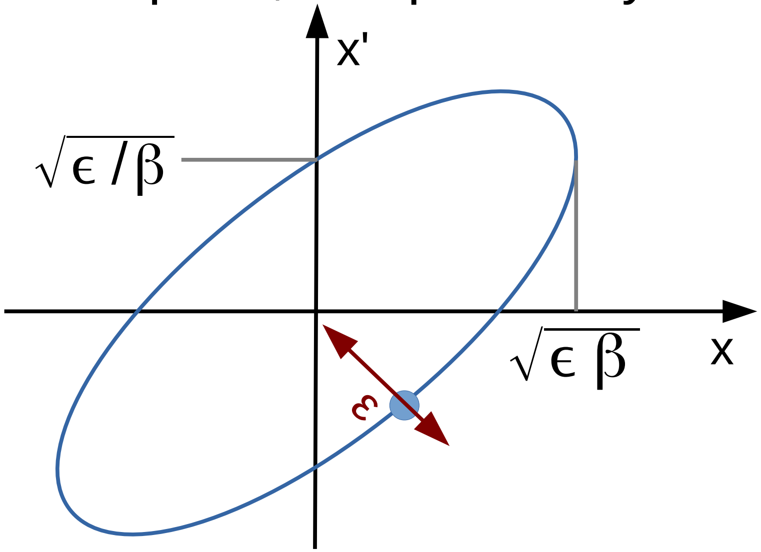
- Particles with same ε are on an ellipse in (x,x') phase space, independently from φ_0 (ellipse changes with s!)
- ε – Courant-Snyder invariant → **area of ellipse**. ~emittance, product of two half-axes. Area of ellipse: επ
- ε constant of motion → ellipse of particles with same ε has **same area along entire ring** (Liouville: constancy of phase space area)



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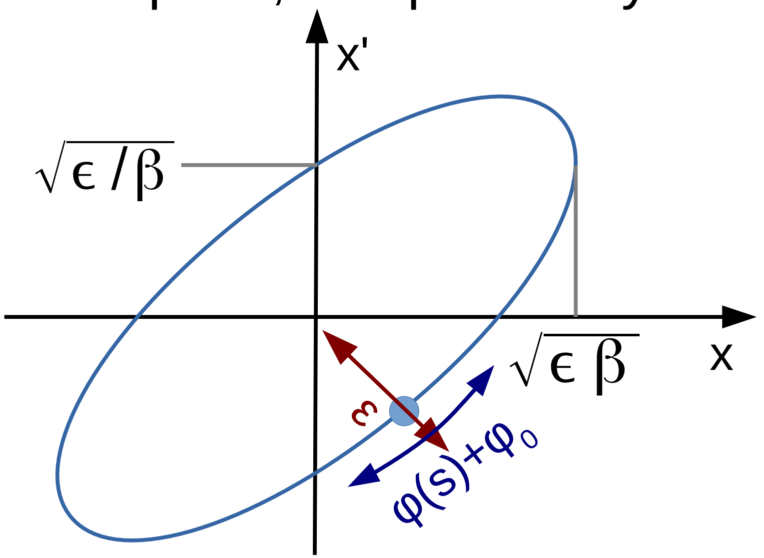
$$\gamma(s)x^2 + 2\alpha(s)x x' + \beta(s)x'^2 = \begin{pmatrix} x & x' \end{pmatrix}\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = \epsilon$$

- Particles with same ε are on an ellipse in (x,x’) phase space, independently from φ_0 (ellipse changes with s!)
- A $\varphi(s)+\varphi_0$ determines position on the ellipse



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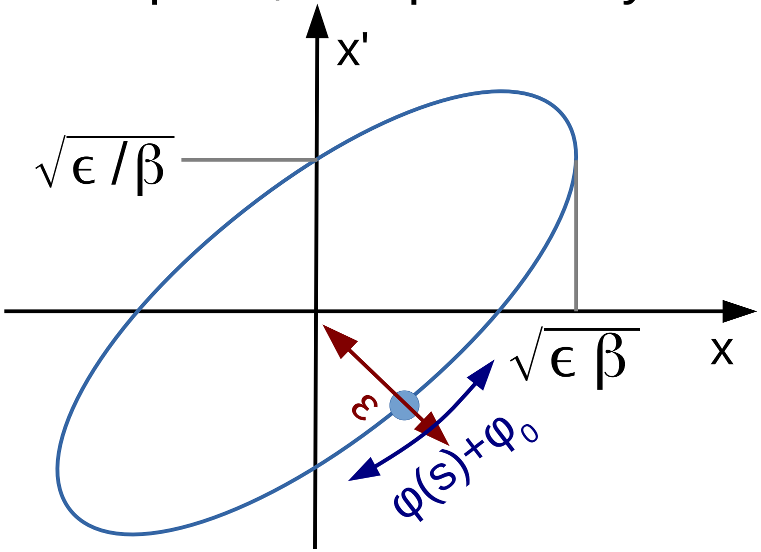

To avoid misunderstandings: this ellipse is NOT a 2D projection of the beam (x,y), but the phase-space distribution (x,x’) or (y,y’)

Twiss parameters

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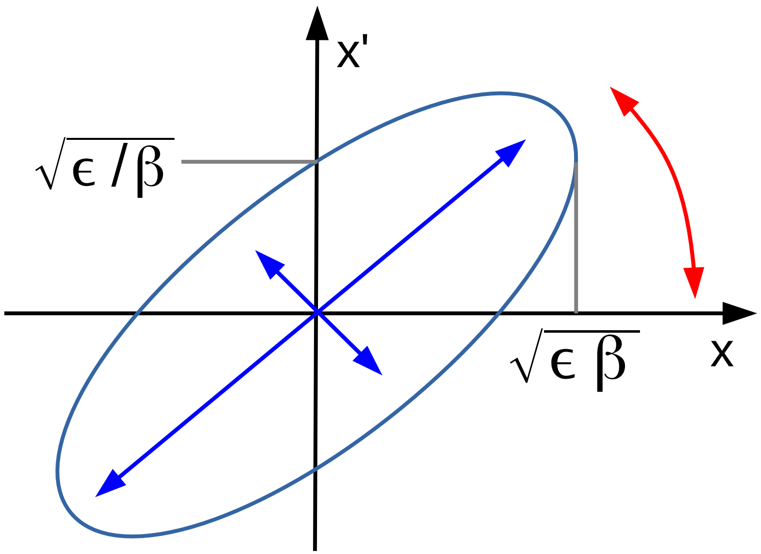
- α, β – **Twiss parameters**, they determine direction and shape of the ellipse (ratio of the axes)
- α(s), β(s) are functions of position along ring
- Therefore shape and direction of ellipse changes along ring circumference..
- ... but its area is constant (ellipse of particles with same epsilons)



Betatron oscillation

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\cos[\phi(s)+\phi_0]$$

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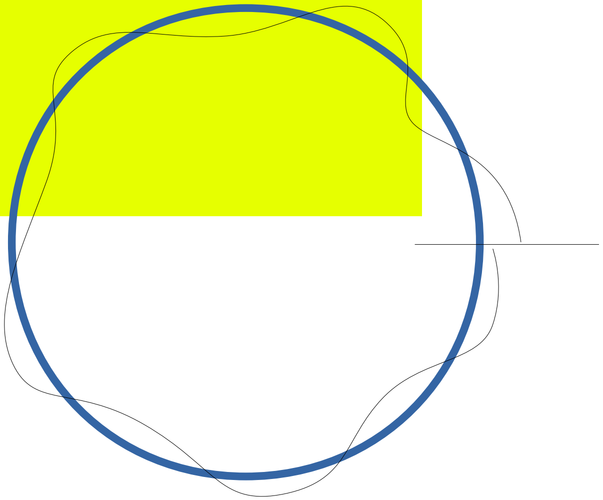
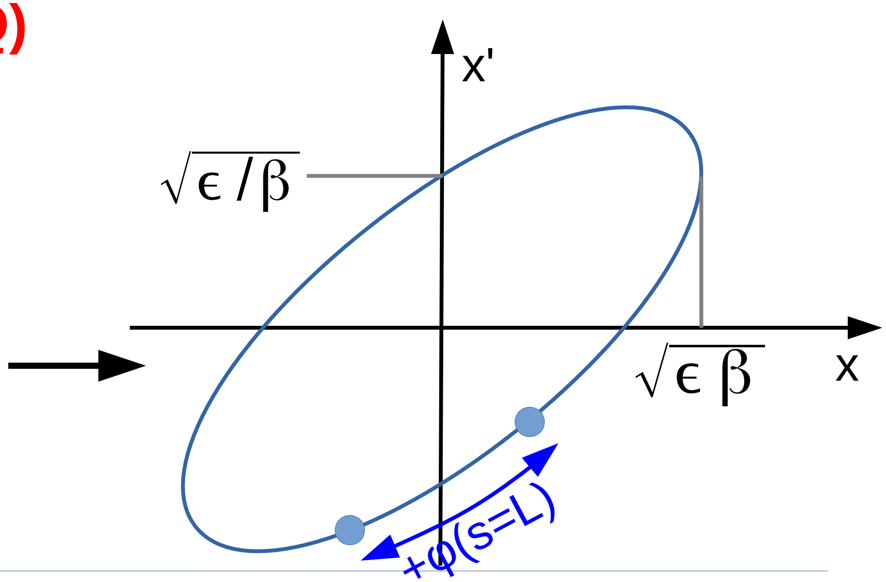


- Motion of a **single particle**:
  ε, φ_0 fix, $\varphi(s)$ changes along ring:
  betatron oscillation
- Number of oscillations in one turn: **TUNE (Q)**
- At every point along the ring, (x,x') is on the ellipse **of that particular position**
- After a complete turn: particle is on the same ellipse
- A single particle traces the ellipse over many turns

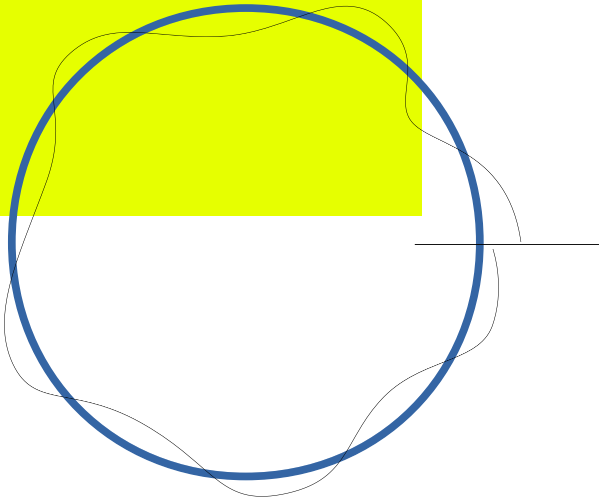


Betatron oscillation

$$x(s) = \sqrt{\beta(s)}\sqrt{\epsilon}\cos[\phi(s)+\phi_0]$$

$$x' = -\sqrt{\frac{\epsilon}{\beta(s)}}\sin[\phi(s)+\phi_0] - \alpha(s)\sqrt{\frac{\epsilon}{\beta(s)}}\cos[\phi(s)+\phi_0]$$

$$\alpha(s) \equiv -\beta'(s)/2 \qquad \gamma(s) \equiv \frac{1+\alpha^2}{\beta(s)}$$



- Motion of a **single particle**:
  ε, φ_0 fix, $\varphi(s)$ changes along ring:
  betatron oscillation
- Number of oscillations in one turn: **TUNE (Q)**
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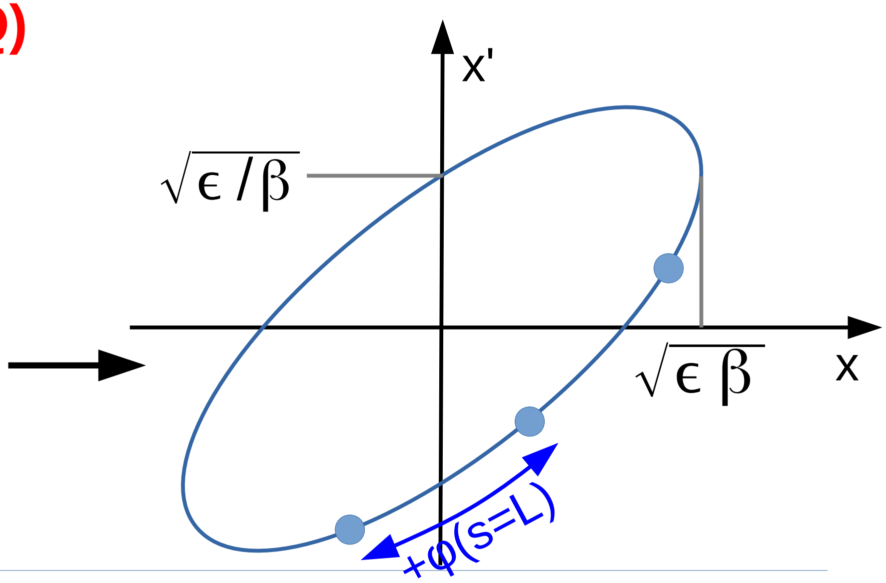

Betatron oscillation

$$x(s)=\sqrt{\beta(s)}\sqrt{\epsilon}\,\cos[\phi(s)+\phi_0]$$

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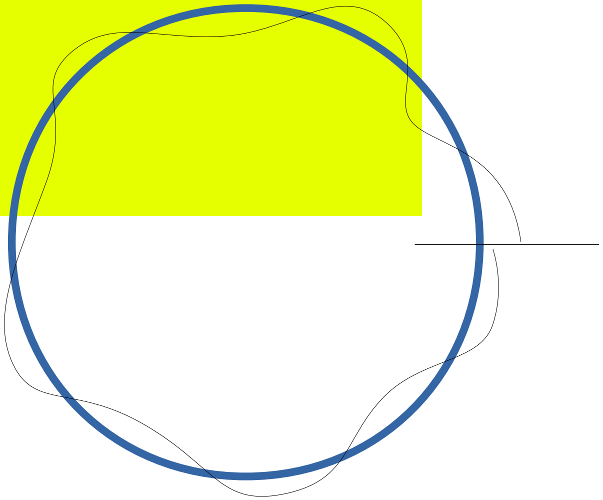

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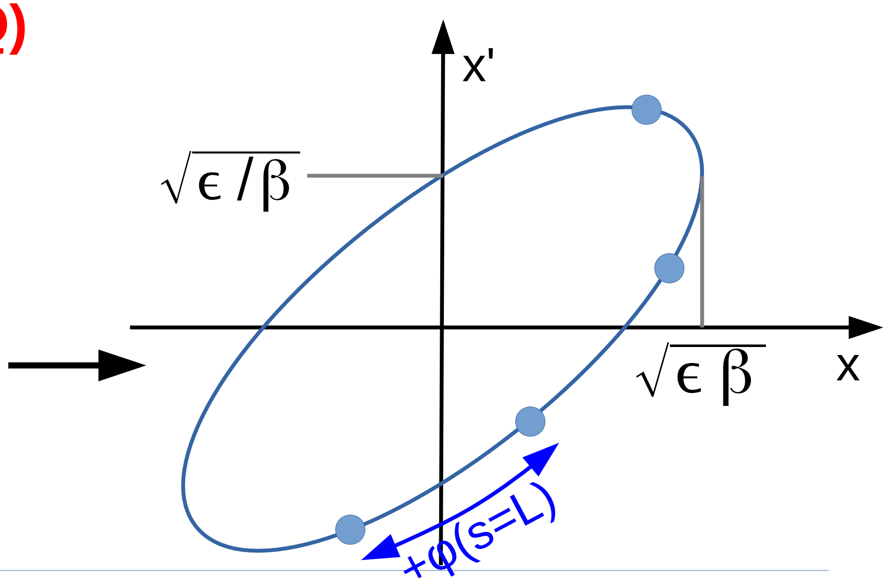

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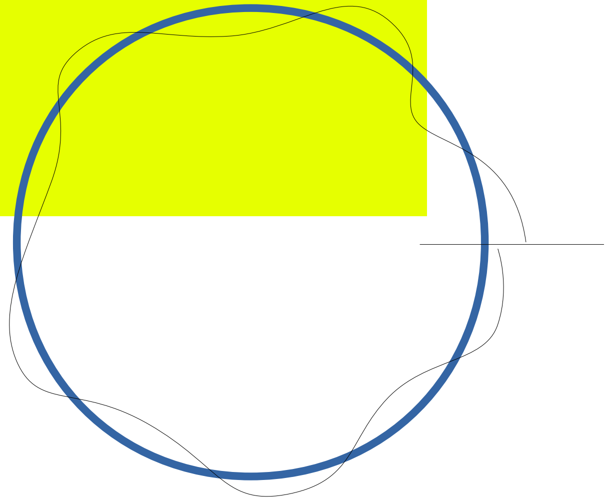

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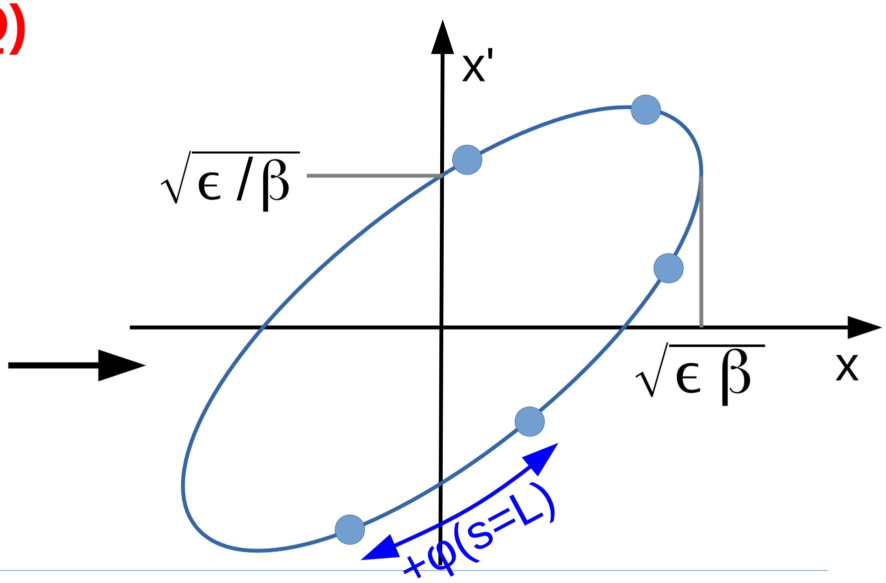

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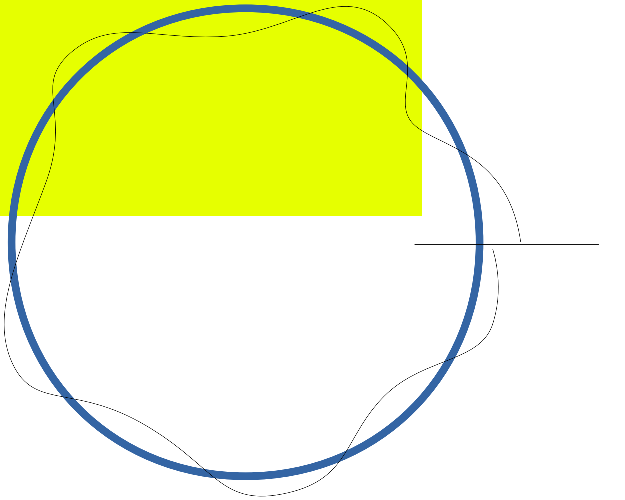

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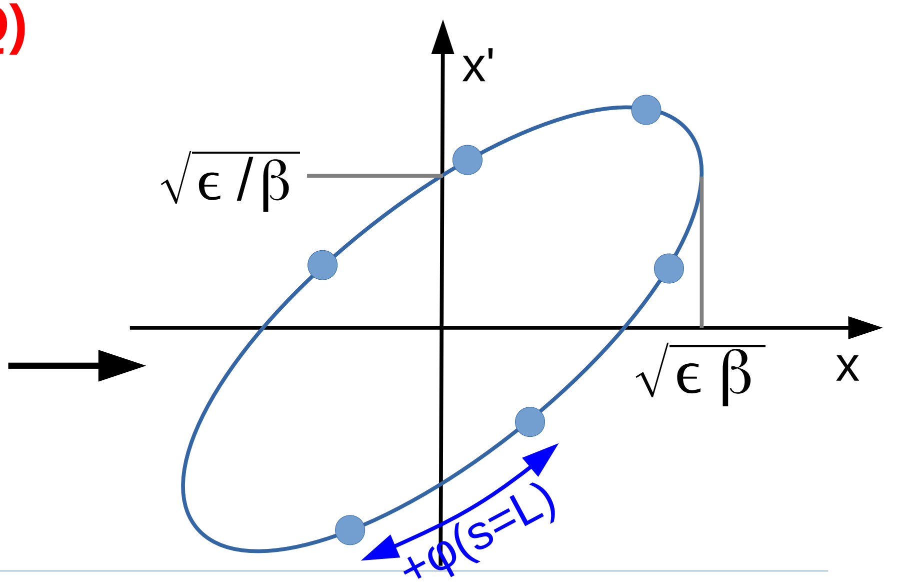

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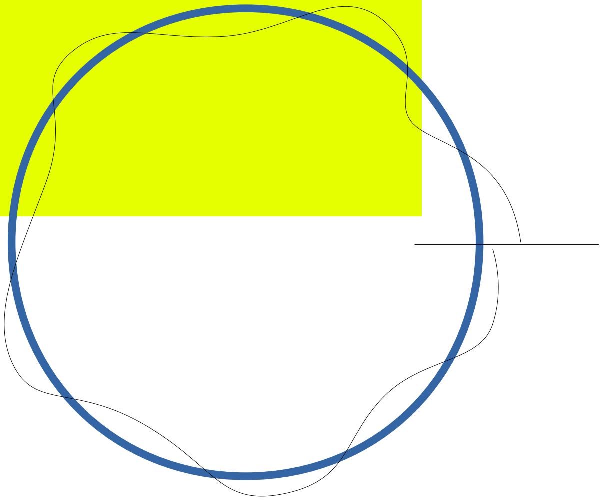

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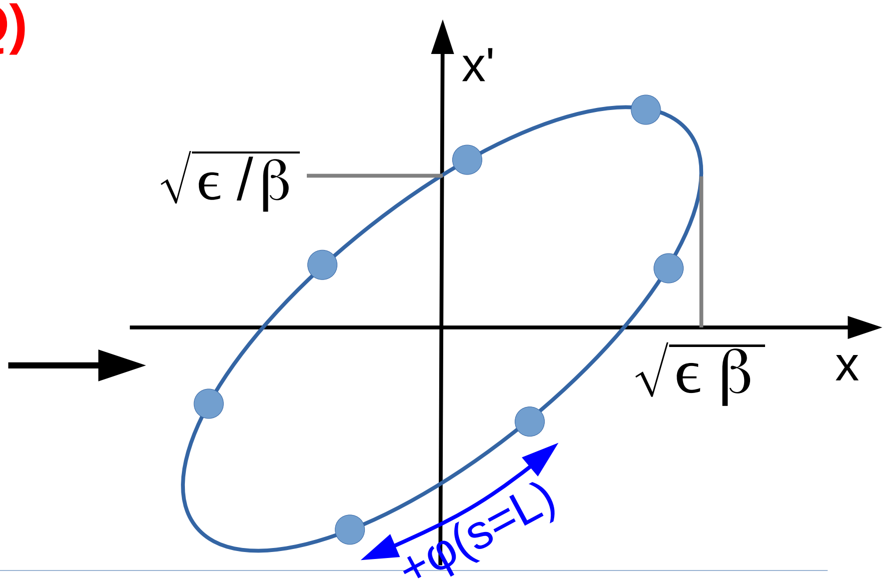

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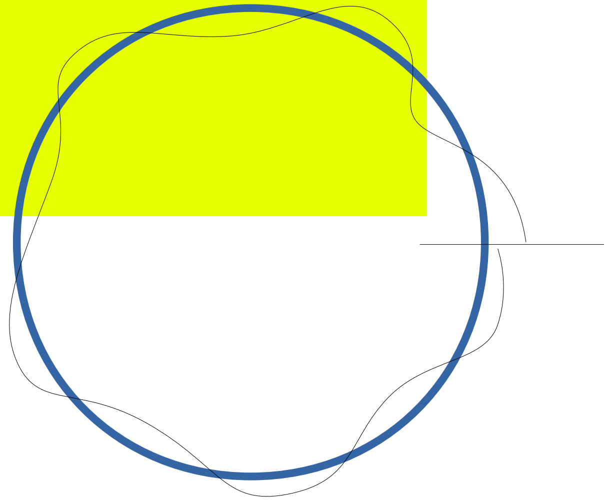

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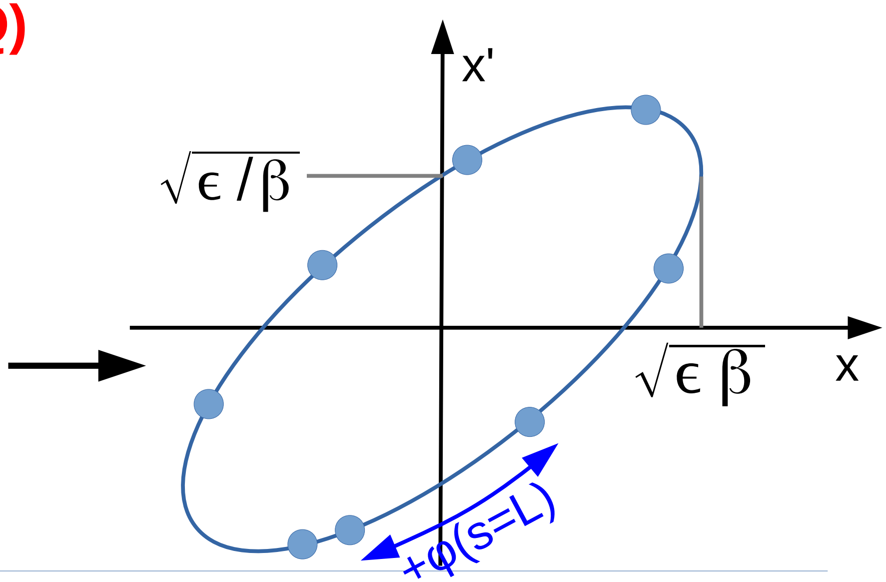

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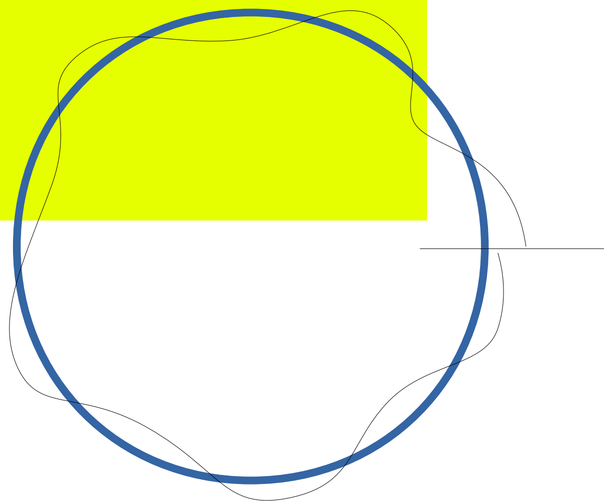

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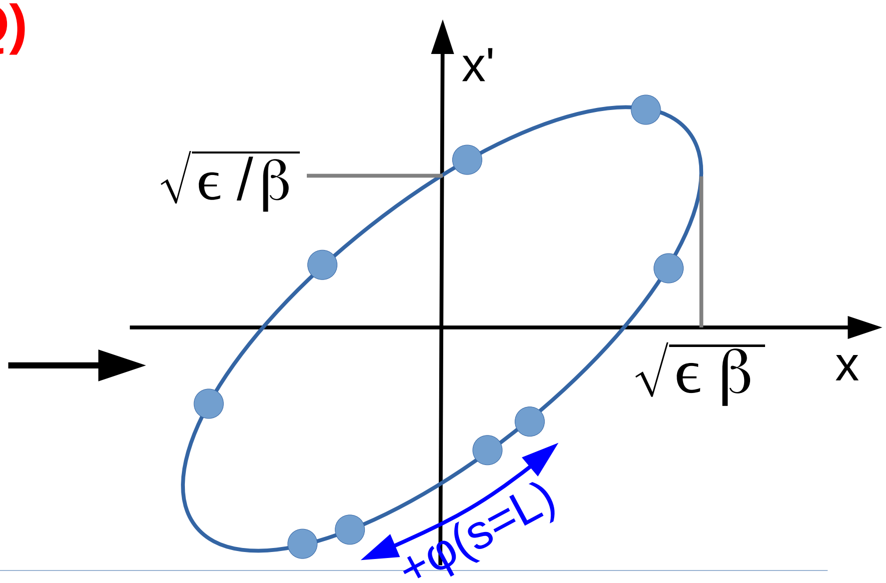

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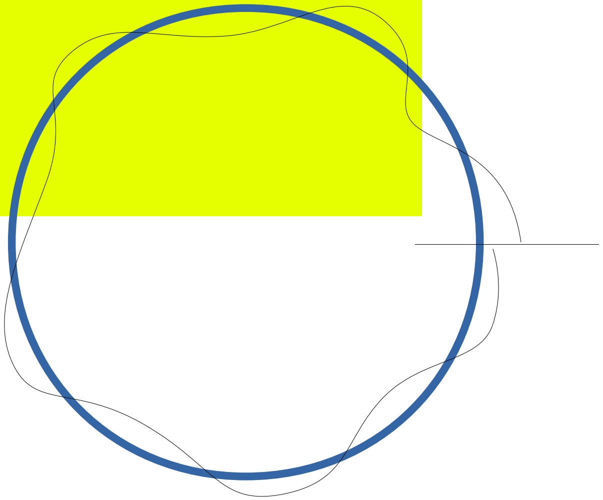

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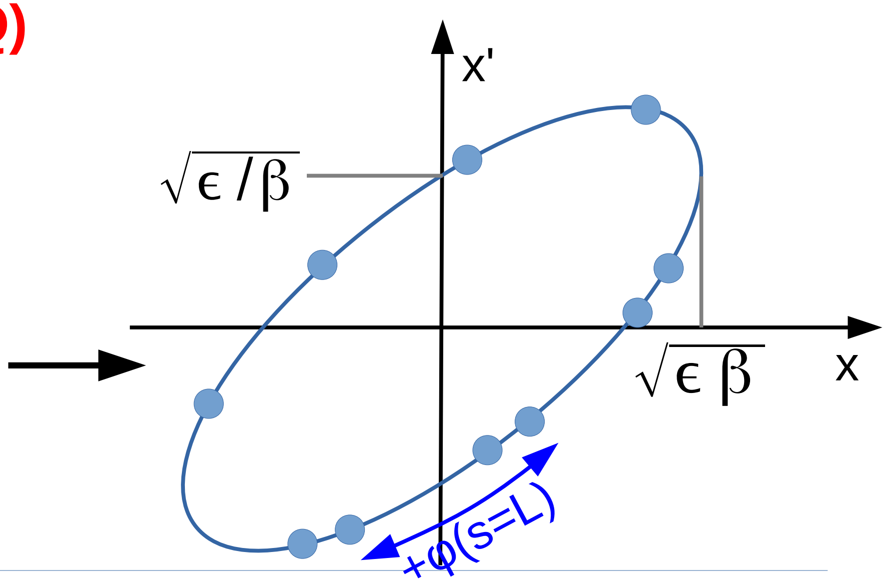

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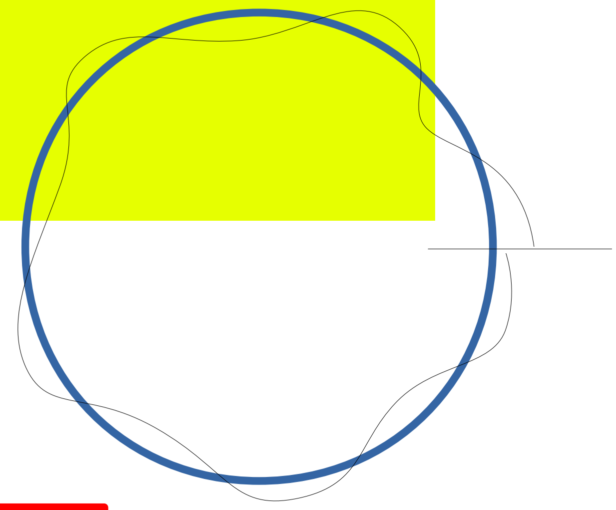
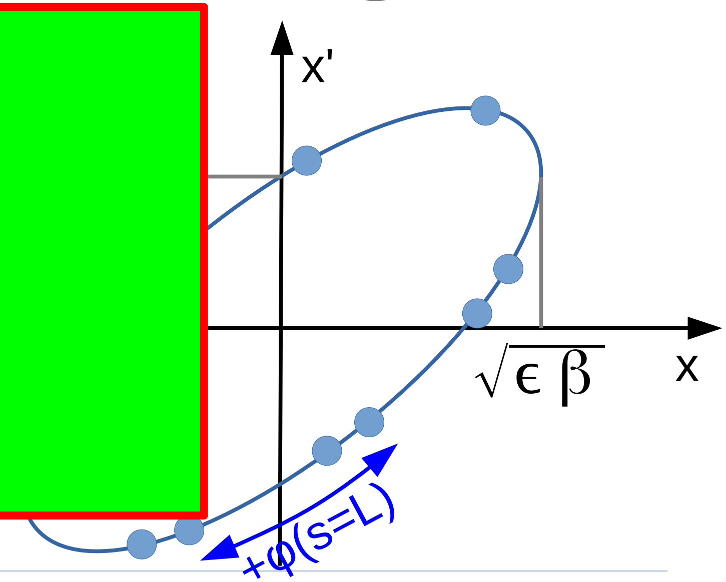


- Motion of a **single particle**: ε, φ_0 fix, $\varphi(s)$ changes along ring: **betatron oscillation**
- Num
- At ev ellips
- After same
- A sin many

- In a homogeneous field, what is the
  - horizontal tune?
  - vertical tune?
- In a weakly focusing ring, what is the
  - horizontal tune?



Invariant ellipse

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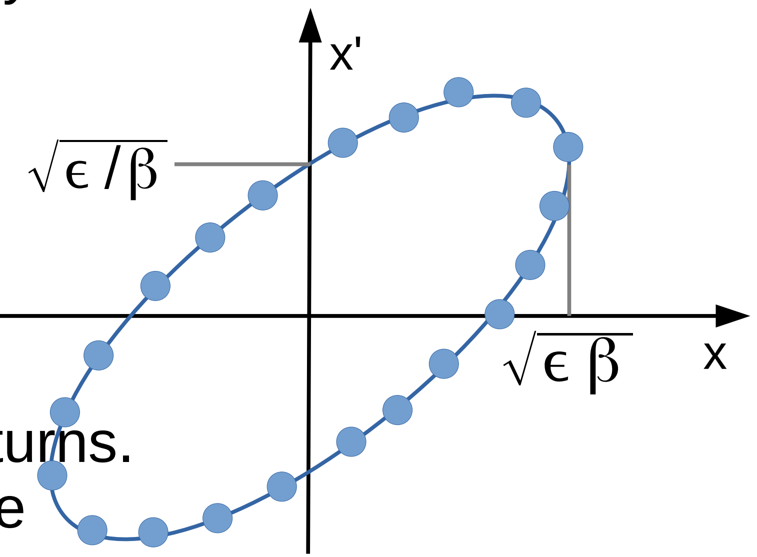
$$\alpha(s) \equiv -\beta'(s)/2 \qquad \gamma(s) \equiv \frac{1+\alpha^2}{\beta(s)}$$

- Take **many particles** initially on an ellipse with a given ε. Their initial phase is $0<\varphi_0<2\pi$
- All of them will be on the same ellipse after any number of complete turns
- Distribution of a real beam can be anything (non-ellipse, or an ellipse different from the Twiss-ellipse)
- Twiss-parameters describe the ellipse at each position, which is **invariant** to any number of turns. Such a beam does not rotate, does not pulsate



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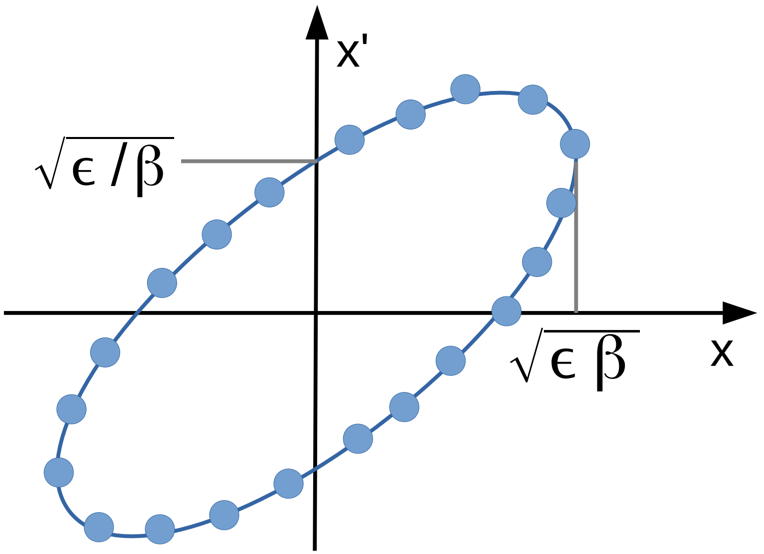
- Take **many particles** initially on an ellipse with a given ε. Their initial phase is $0<\varphi_0<2\pi$
- At every point s: $-\sqrt{\beta(s)\epsilon}<x(s)<\sqrt{\beta(s)\epsilon}$
- $\sqrt{\beta(s)\epsilon}$ is therefore the **envelope** of trajectories with a given ε



Matched beam

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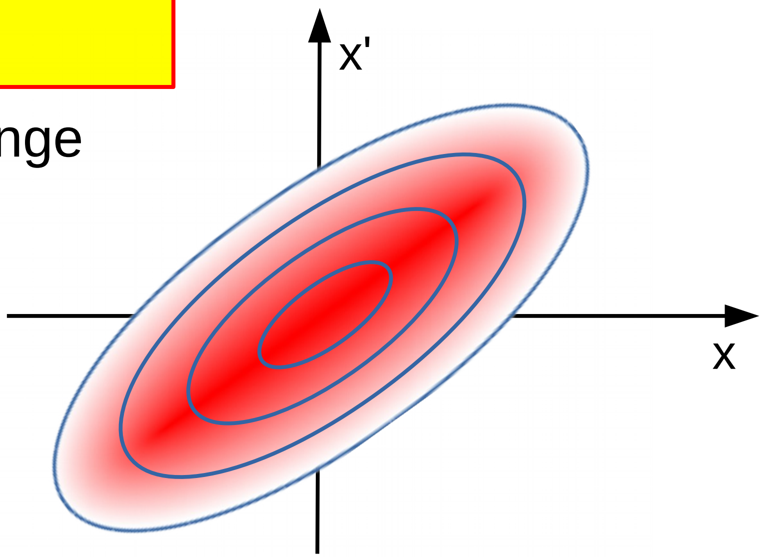
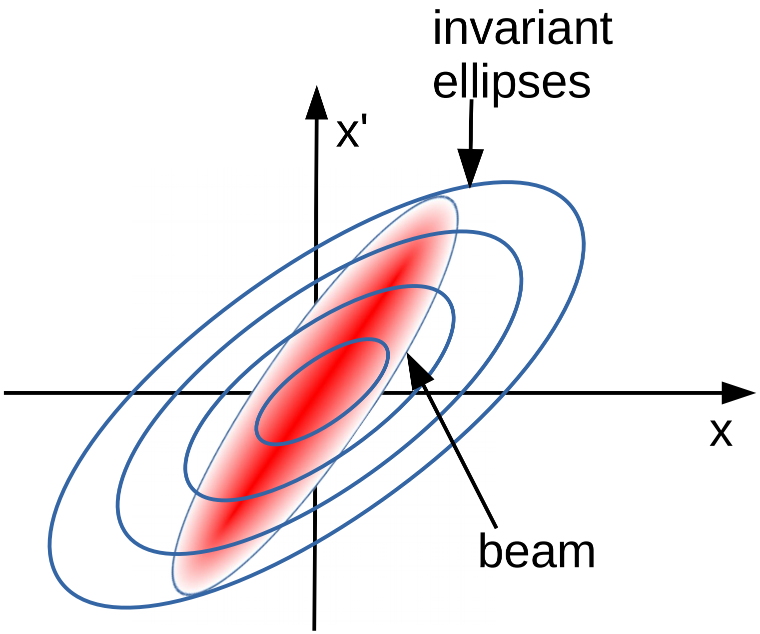
- A **real beam** has typically roughly 2D (Gaussian) distribution in phase space (with ellipses as equipotentials)
- **IF** this 2D distribution coincides with the Twiss ellipse, we then have a
  MATCHED BEAM
- For a matched beam, beam profile does not change with time at any position along the ring
- Any particle will remain on similar ellipses
- **Emittance of beam**: area of ellipse containing a given fraction (given in σ) of the beam
  typical unit: π mm mrad



Mismatched beam

- For an **unmatched beam** (i.e. shape of beam does not coincide with Twiss-ellipse): **beam shape is different** after each turn



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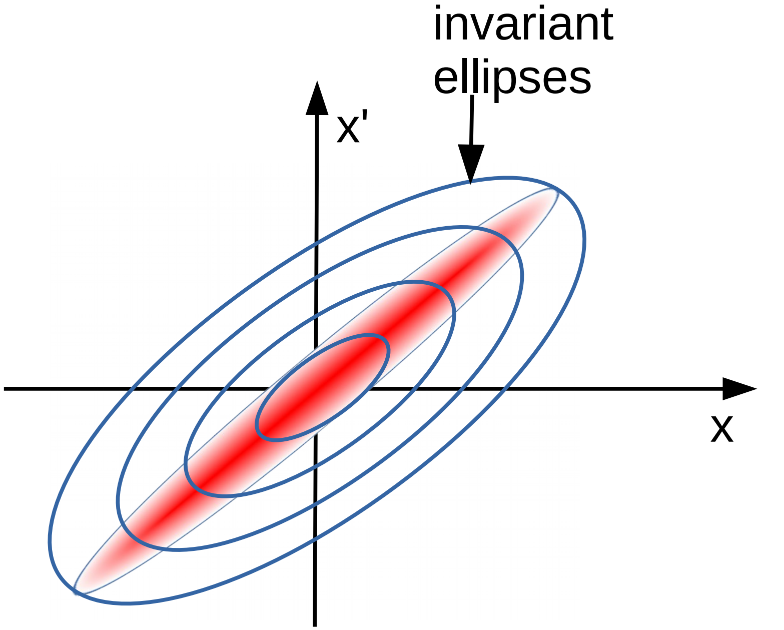

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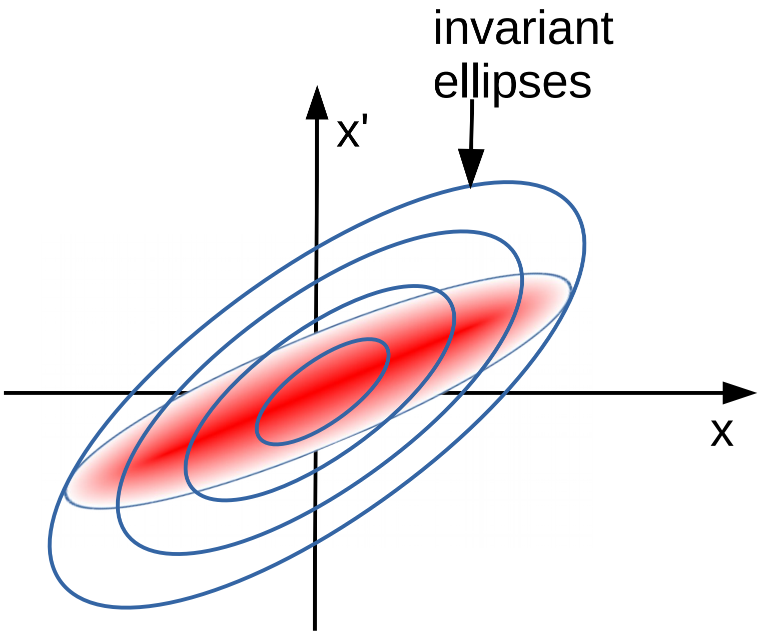

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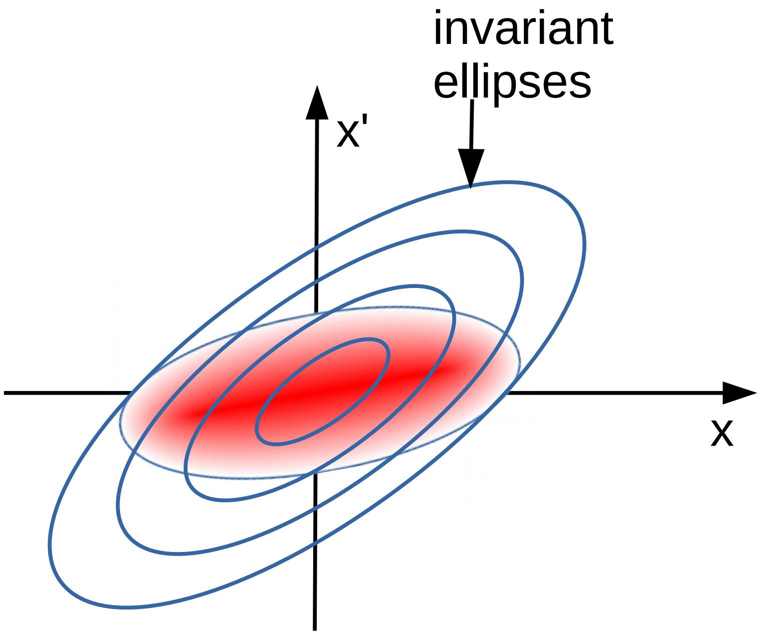

Mismatched beam

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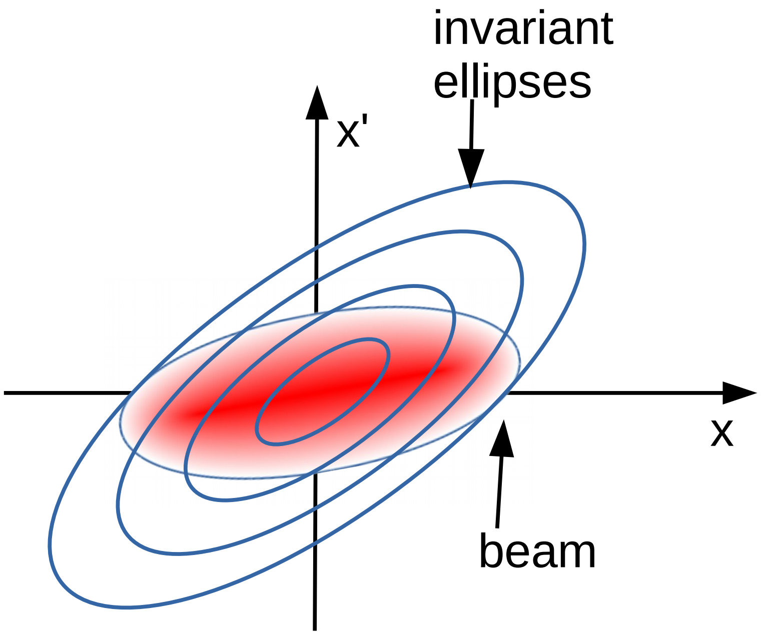
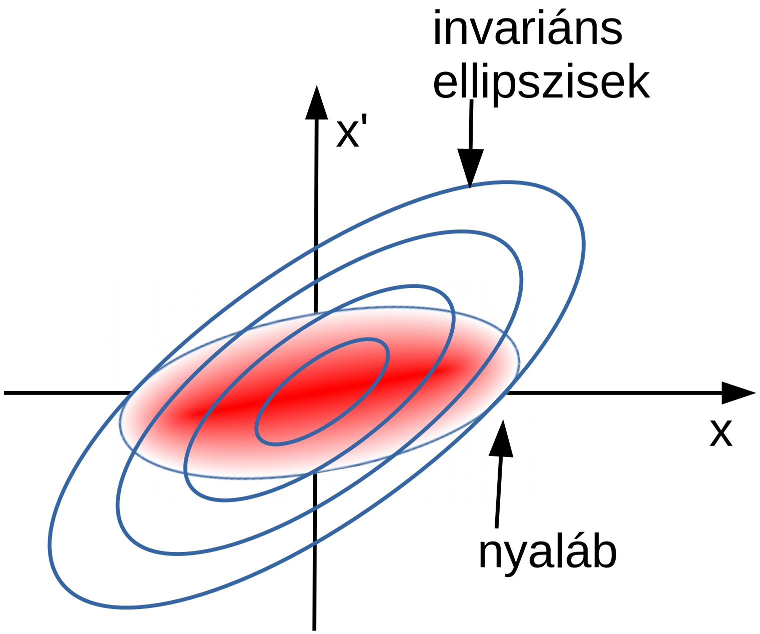
Area of the real ellipse of the beam is constant, but it changes shape

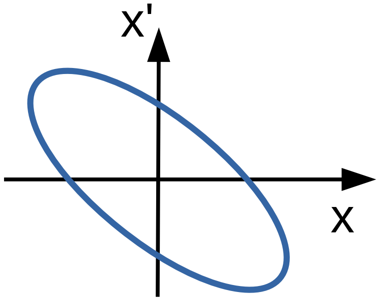


Mismatched beam

- For an **unmatched beam** (i.e. shape of beam does not coincide with Twiss-ellipse): **beam shape is different** after each turn
- Beam size measured at a fixed position will oscillate
- Over many turns these ellipses will paint the invariant ellipse, which envelops the initial distribution
- This ellipse has a larger area than the beam distribution
- Time-averaged emittance and beamsize larger than necessary
- Filamentation: due to nonlinear effects the beam distribution will filament (perimeter rotates slower, for example), and effectively fills the whole invariant ellipse not only in time-average



Interpretation of the Twiss ellipse, its evolution



Diverging (after the waist), or converging (proceeding towards a waist) beam?

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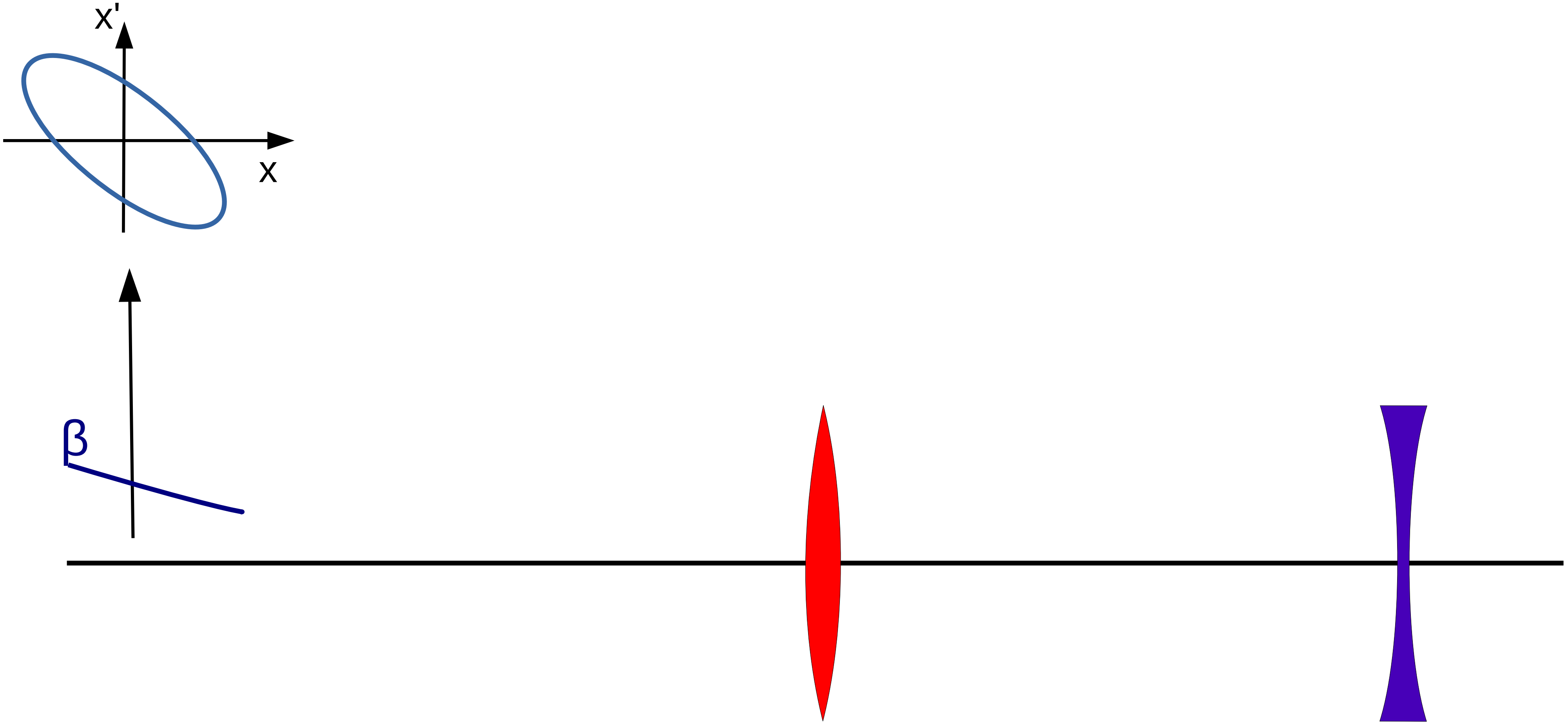

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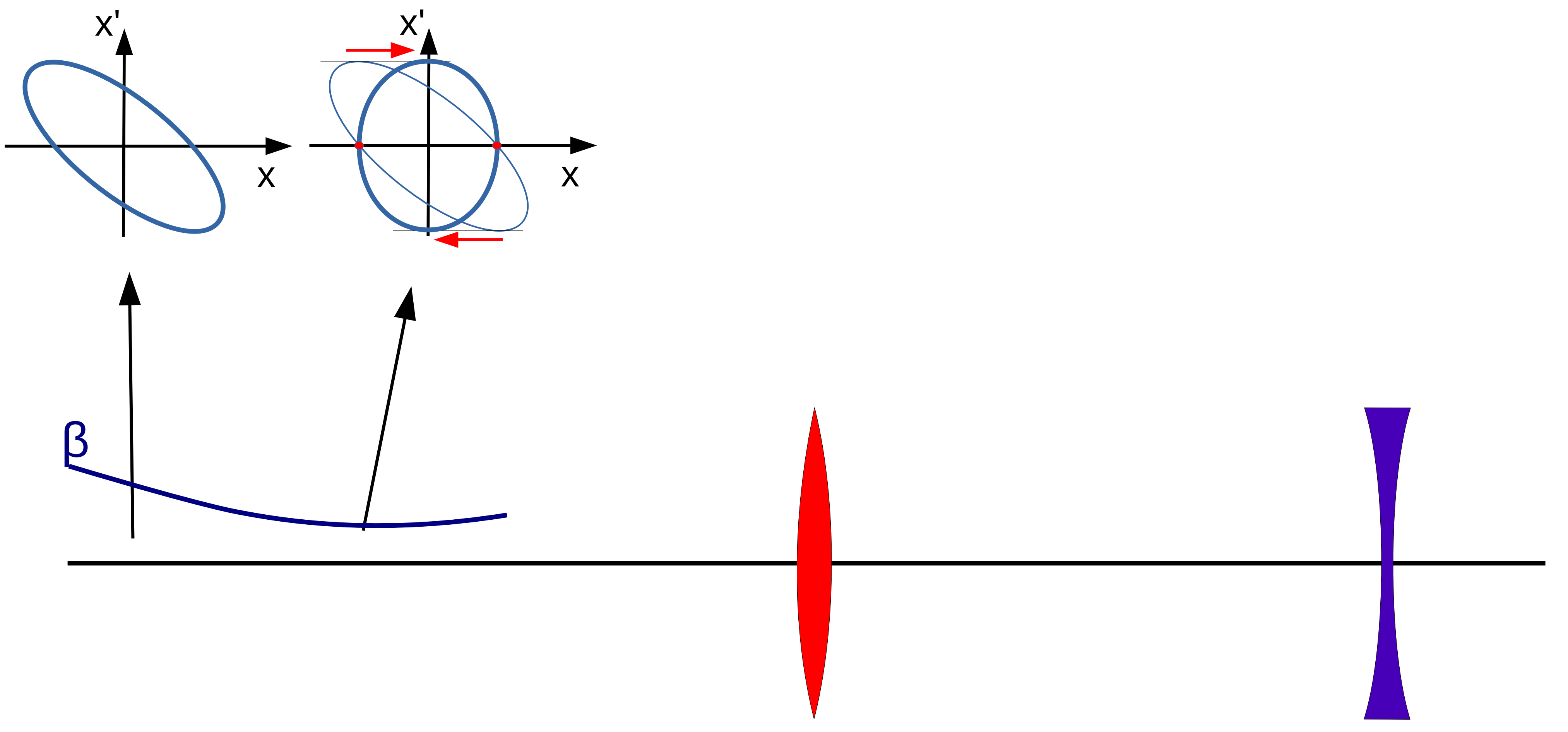

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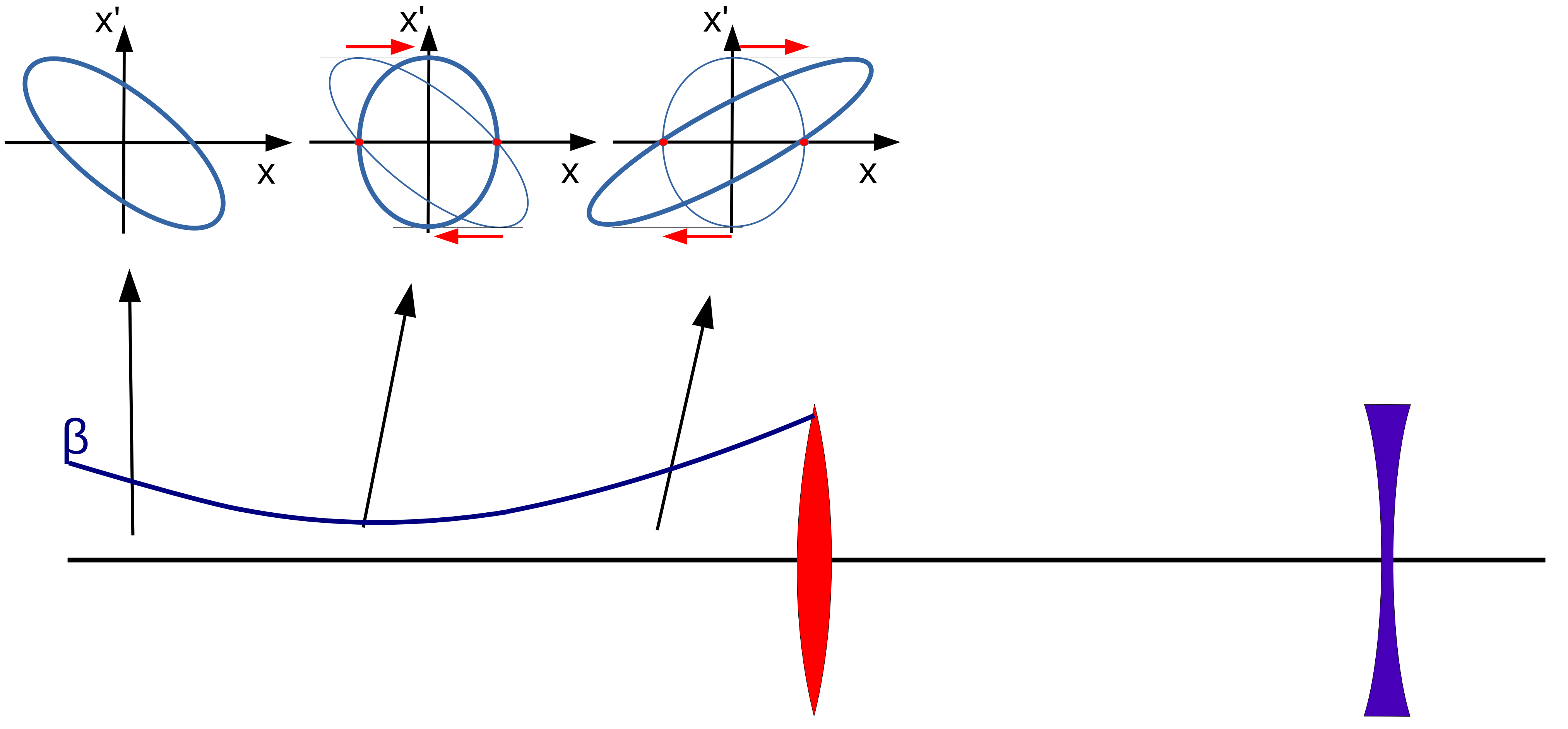

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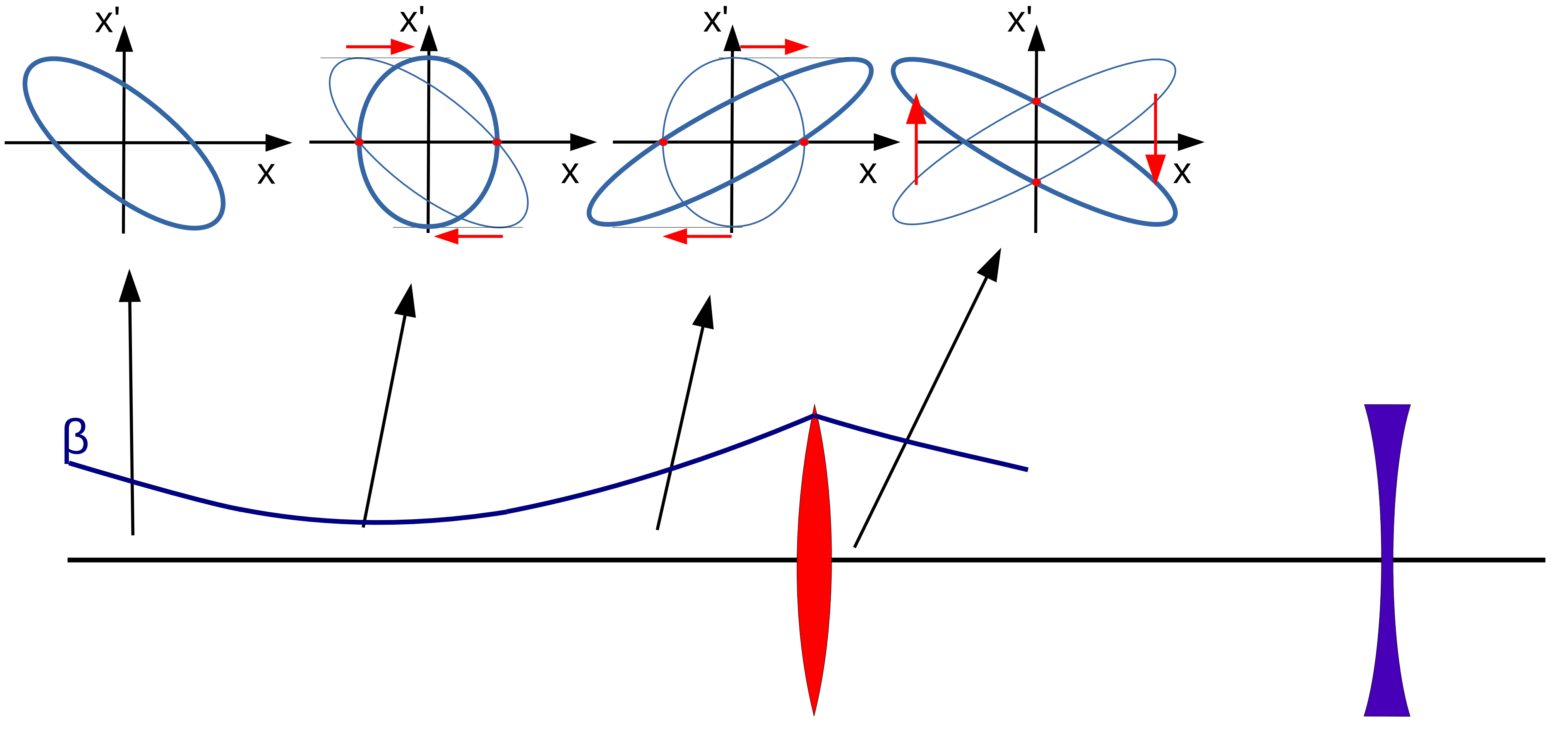

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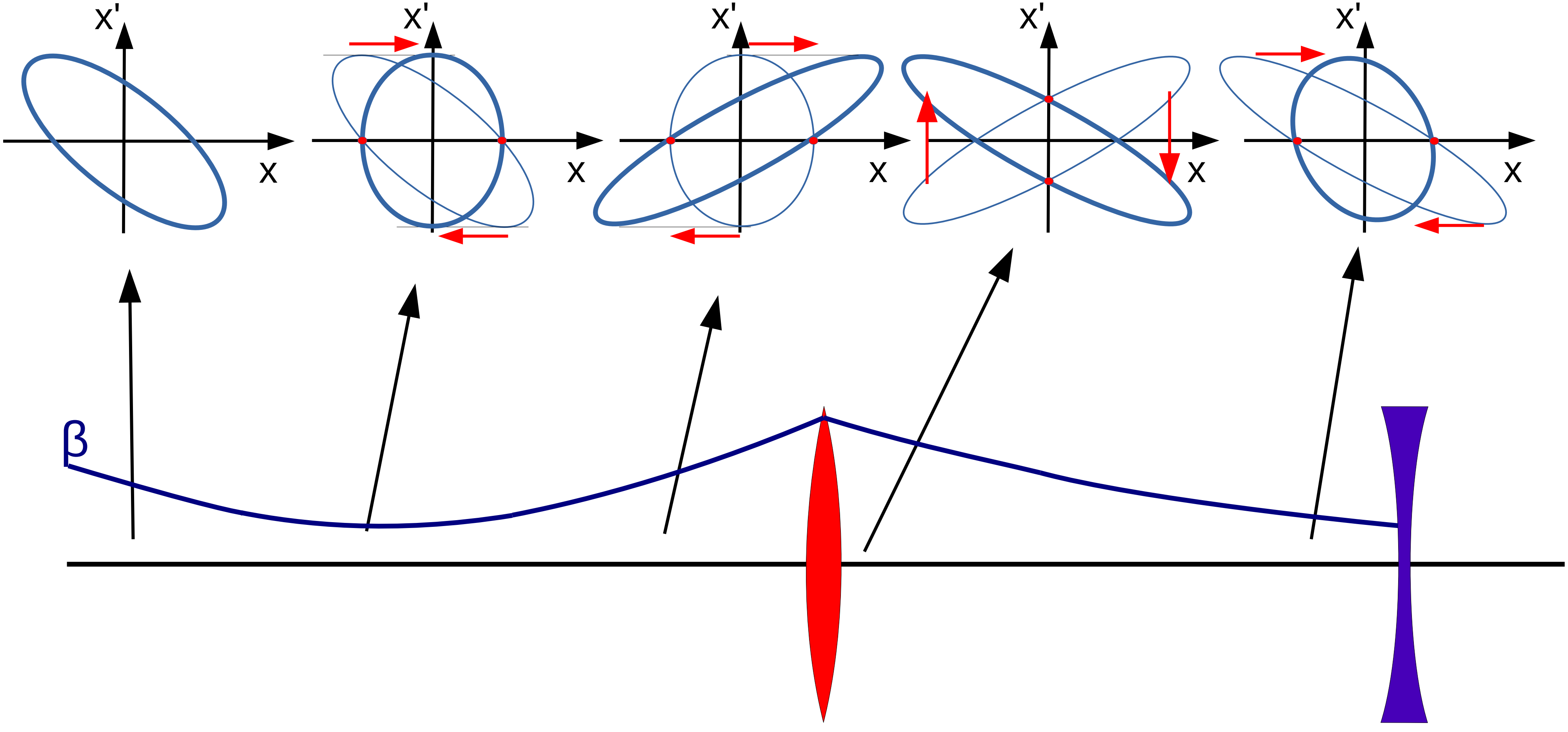

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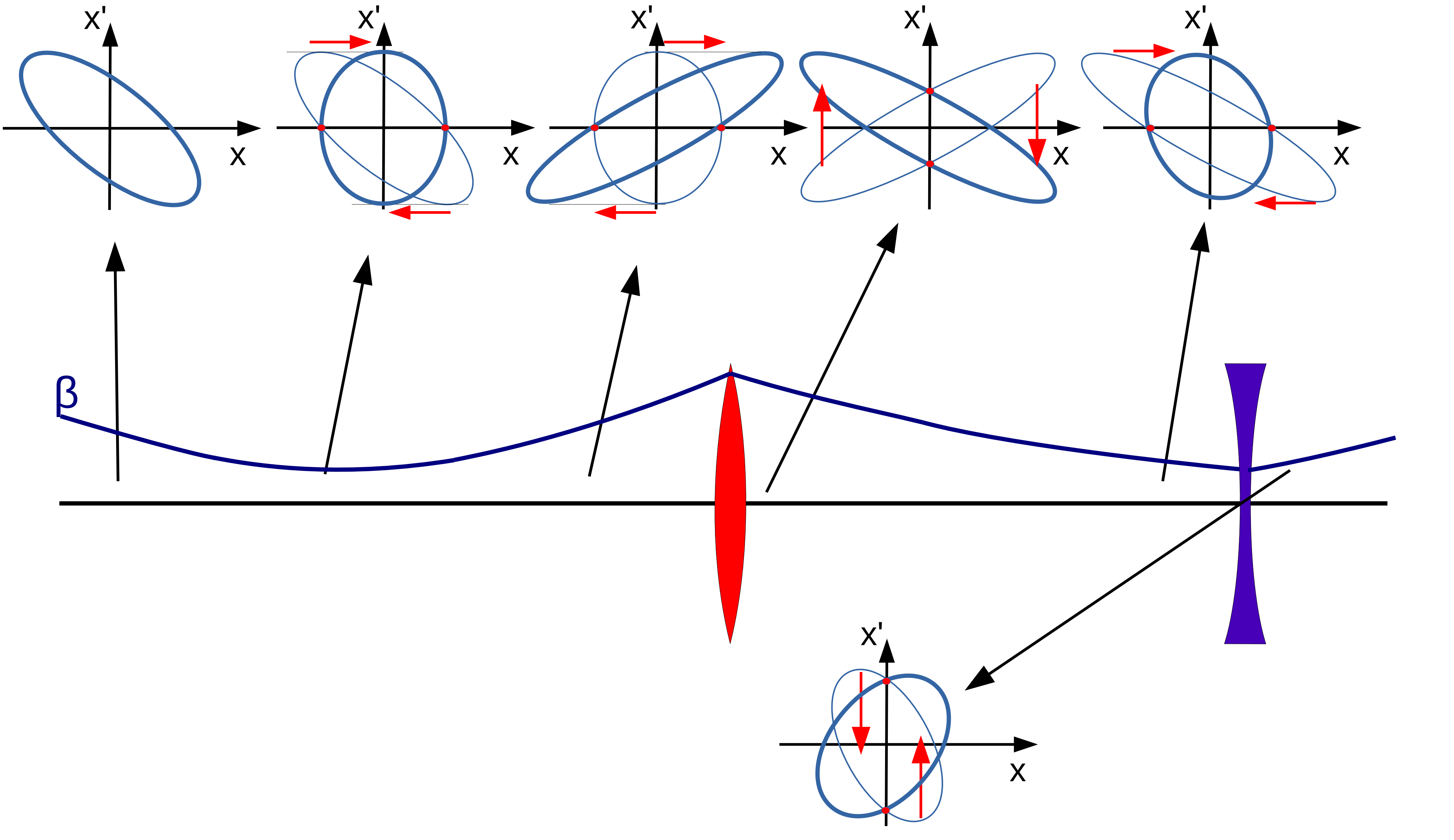

Transverse dynamics: ε & emittance

- ε
  - Constant of motion of a single particle (Courant-Snyder invariant)
  - Parameter of a Twiss-ellipse (together with α és β) – at any point around the ring(s) a particle with a given ε will be on the ellipse determined by α(s), β(s) and ε
- Beam emittance (only meaningful if the fraction of beam is also mentioned: how many sigma?)
  - Area (or product of two half-axes, depending on convention) of Twiss-ellipse, which contains x-sigma of the beam
  - emittance = ε π vagy ε (depending on convention)
  - For example: 95% vertical emittance = 10 π mm mrad →
    - Area of ellipse which contains 95% of beam in (y,y') phase space = 10 π mm mrad
  - Or 95% vertical emittance ε = 10 mm mrad →
    - Product of two half axes of ellipse containing 95% of beam in (y,y') phase space is 10 mm mrad

Beamsize around the collision point

- Due to the detectors around the collision points, no quadrupole magnets can be placed nearby
- For a relatively large distance (~10 m)
- In order to have a large number of collisions, need very small beamsize
- β* (β function's value at the collision point) must be made as small as possible
- In a drift space: $\beta(s) = \beta_0 + s^2/\beta_0$ $\quad (\beta_0 = \beta^*)$
- Since β* is small, beam diverges very fast (Liouville: if beam is compressed in x, must increase in x', i.e. have large divergence)
- Nearest quads are relatively far away
- They need large apertures

Transverse dynamics: transfer matrix

- Solutions of a 2nd order linear diff. equation form a 2D linear space
- All solution can be written as the linear combination of two linearly independent solutions $x_1(s)$, $x_2(s)$

$x(s) = a*x_1(s) + b*x_2(s)$
$x'(s) = a*x_1'(s) + b*x_2'(s)$

- Useful to choose this basis: $x_1(0) = 1$, $x_1'(0)=0$; $x_2(0) = 0$, $x_2'(0)=1$
- In this basis:

Transfer matrix between $0 \rightarrow s$

$$\begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \equiv M(s,0) \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

- If two well-chosen trajectories are known along the ring (for example we can simulate) → we know all trajectories!
- Transfer matrix between any two points on the ring can be calculated from these two trajectories
- Simple matrix-multiplication to step from point to point

$$M(s_2, s_0) = M(s_2, s_1) \cdot M(s_1, s_0)$$

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- Transfer matrix between any two points on the ring can be calculated fro

Determinant of the transfer matrix is 1!
(Liouville's theorem)

- Simple matrix-multiplication to step from point to point

$$M(s_2, s_0) = M(s_2, s_1) \cdot M(s_1, s_0)$$

Transverse dynamics: transfer matrix

- In practice:
  - Transfer matrix of the ring's components is known a priori
    - quadrupoles (lense)
    - dipole bending magnets
    - effect of fringe fields
    - drift space (no magnetic field)
  - We use these matrices to construct the ring optics
- Matrix optics is the most important first technique when designing a ring
  - are the trajectories stable?
  - what is the maximum beam size
- For uncoupled motions →
- Coupling can be caused:
  - mis-oriented quadrupoles
  - sextupole magnets
  - solenoid magnets
  - field errors

$$\begin{pmatrix} x_2 \\ x'_2 \\ y_2 \\ y'_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{21} & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & M_{34} \\ 0 & 0 & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \end{pmatrix}$$

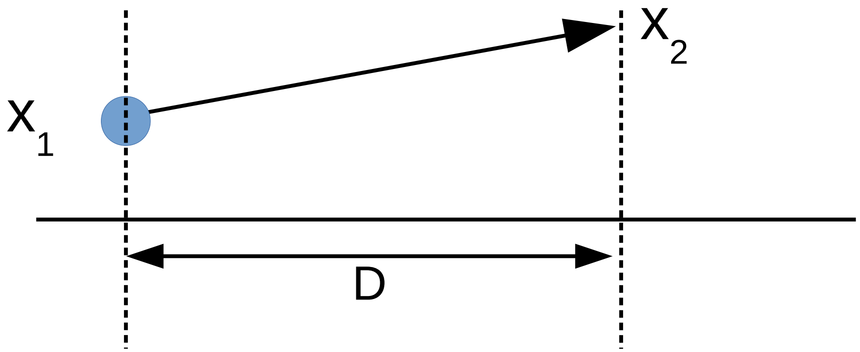
Introducting the transfer matrix without Hill equation

- $x_2 = x(s_2) = F(x_1,x'_1)$
  $x'_2 = x'(s_2) = G(x_1,x'_1)$
- To first order
  $x_2 = F(0,0) + \partial_1 F(0,0)\cdot x_1 + \partial_2 F(0,0)\cdot x'_1$
  $x'_2 = G(0,0) + \partial_1 G(0,0)\cdot x_1 + \partial_2 G(0,0)\cdot x'_1$
- Since the central particle moves on the reference orbit everywhere (x=x'=0 everywhere)
  $F(0,0) = G(0,0) = 0$
- Therefore

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \partial_1 F & \partial_2 F \\ \partial_1 G & \partial_2 G \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Transfer matrix: drift space

- Drift space = no magnetic field



$$x_2 = x_1 + D \cdot x'_1$$
$$x'_2 = x'_1$$

- Transfer matrix:

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Transfer matrix: hard-edge long quadrupole

- x'' = k x
- Assumption (hard edge): k=const within the quad: 0<s<L, and then drops to zero abruptly
- Solution

$$x = A\cos(\sqrt{k}\,s) + B\sin(\sqrt{k}\,s) \qquad \text{Focusing plane}$$
$$x = A\cosh(\sqrt{k}\,s) + B\sinh(\sqrt{k}\,s) \qquad \text{Defocusing plane}$$

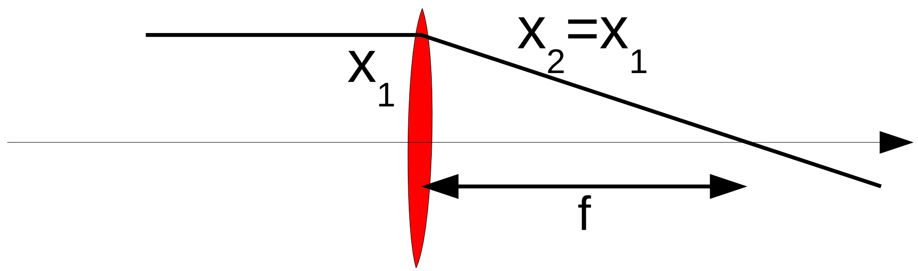
- Corresponding transfer matrix:

$$T_F = \begin{pmatrix} \cos(L\sqrt{k}) & \frac{1}{\sqrt{k}}\sin(L\sqrt{k}) \\ -\sqrt{k}\sin(L\sqrt{k}) & \cos(L\sqrt{k}) \end{pmatrix}$$

$$T_D = \begin{pmatrix} \cosh(L\sqrt{k}) & \frac{1}{\sqrt{k}}\sinh(L\sqrt{k}) \\ \sqrt{k}\sinh(L\sqrt{k}) & \cosh(L\sqrt{k}) \end{pmatrix}$$

Transfer matrix: short lense (quadrupole)

- Thin lense = much shorter than oscillation wavelength, displacement neglected within the lens
- Receives only a “kick”, proportional to coordinate:



$$x_2 = x_1$$
$$x'_2 = x'_1 + F\, x_1$$

- $x'_2=-x_1/f$, from which the focusing power is: F=-1/f
- Transfer matrix: $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

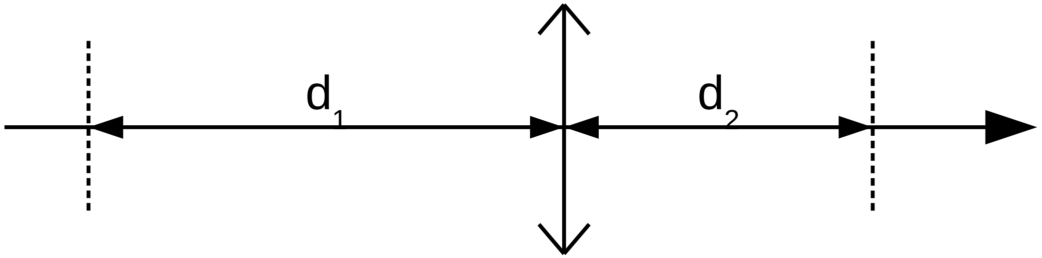
F – focusing power
f – focal distance

- Quadrupole with strength k: F= – k L (where L is the length of the quad)

  (approximation is valid if displacement within quad can be neglected, i.e. f>>L)

Decomposition of transfer matrix

- A 2x2 matrix with a determinant of 1 has three free parameters
- Any 2x2 matrix with a determinant of 1 can be written as a product of 3 well-chosen matrices, for example drift-focus-drift:



$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+d_2 F & d_1+d_2+d_1 d_2 F \\ F & 1+d_1 F \end{pmatrix}$$

$F = m_{21}$ lower-left element is still the focusing strength

$$d_1 = (m_{11} - 1)/m_{21}$$

$$d_2 = (m_{22} - 1)/m_{21}$$

$$d_1 + d_2 = (m_{11} + m_{22} - 2)/m_{21} = [Tr(T) - 2]/m_{21} \quad \text{(effective length)}$$

Stability

- Transfer matrix of a full turn: $$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
- For n turns:
  (v_1, v_2 eigenvectors, λ_1 λ_2 eigenvalues) $$M^n=$$???

Stability

- Transfer matrix of a full turn: $$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- For n turns:
  (v_1, v_2 eigenvectors, λ_1 λ_2 eigenvalues) $$M^n = \lambda_1^n \cdot \vec{v}_1 \circ \vec{v}_1 + \lambda_2^n \cdot \vec{v}_2 \circ \vec{v}_2$$

$$\lambda_{1,2} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - |M|} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - 1} \quad \text{where } T = a + d \quad \text{(trace)}$$

$$\lambda_1 \cdot \lambda_2 = 1$$

- If **T>2**: λ_1 and λ_2 are real, and for one of them $|\lambda|>1$ → **instability**

- If **T<2: stability**

$$\lambda_{1,2} = \frac{T}{2} \pm \sqrt{1 - \left(\frac{T}{2}\right)^2} \equiv \cos\mu \pm i \sin\mu = e^{\pm i\mu} \qquad \cos(\mu) \equiv T/2$$

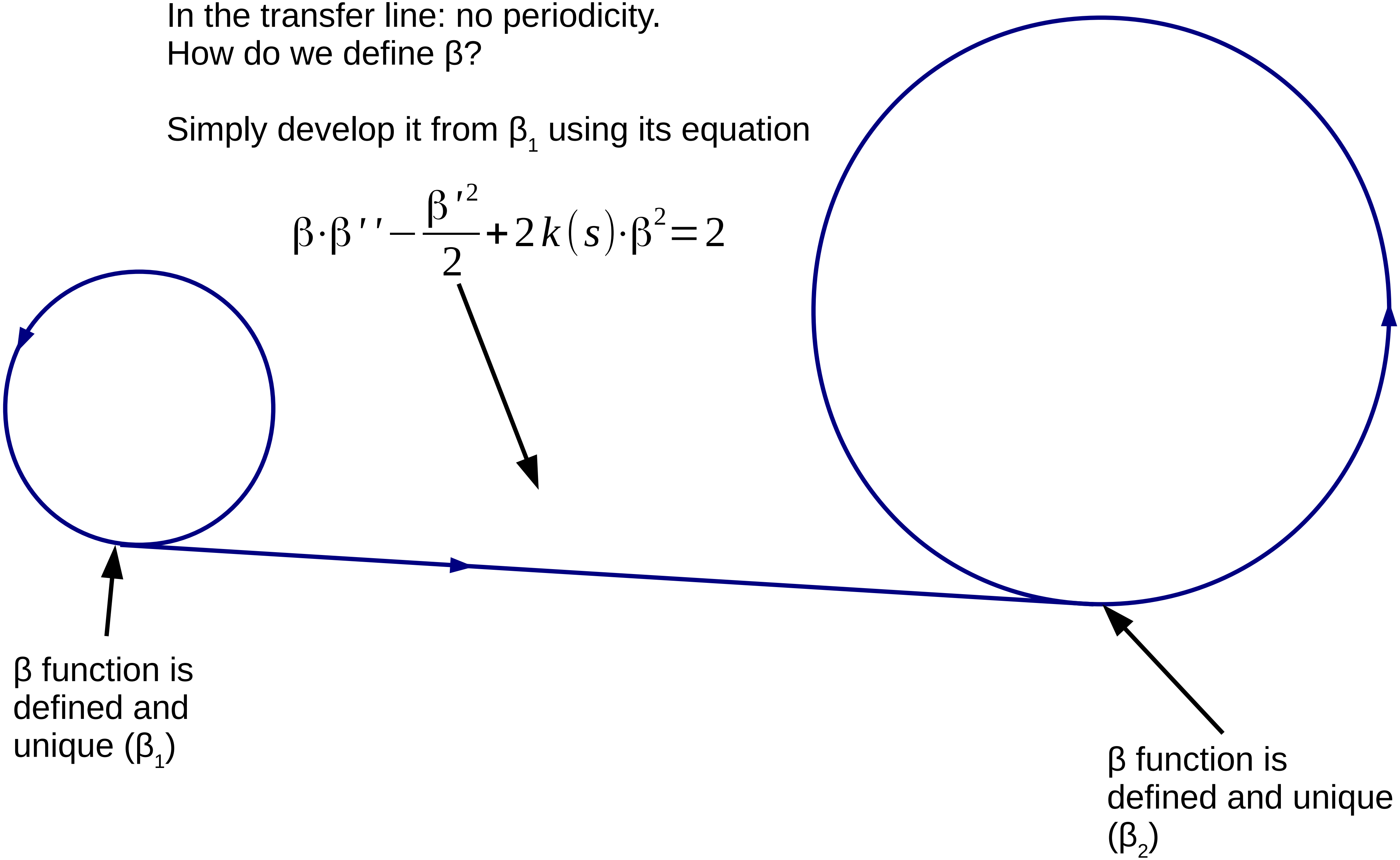
$$M^n = e^{in\mu} \vec{v}_1 \circ \vec{v}_1 + e^{-in\mu} \vec{v}_2 \circ \vec{v}_2$$ real and finite for arbitrary n [homework]

Twiss parameters in a transfer line

In the transfer line: no periodicity.
How do we define β?

Simply develop it from β_1 using its equation

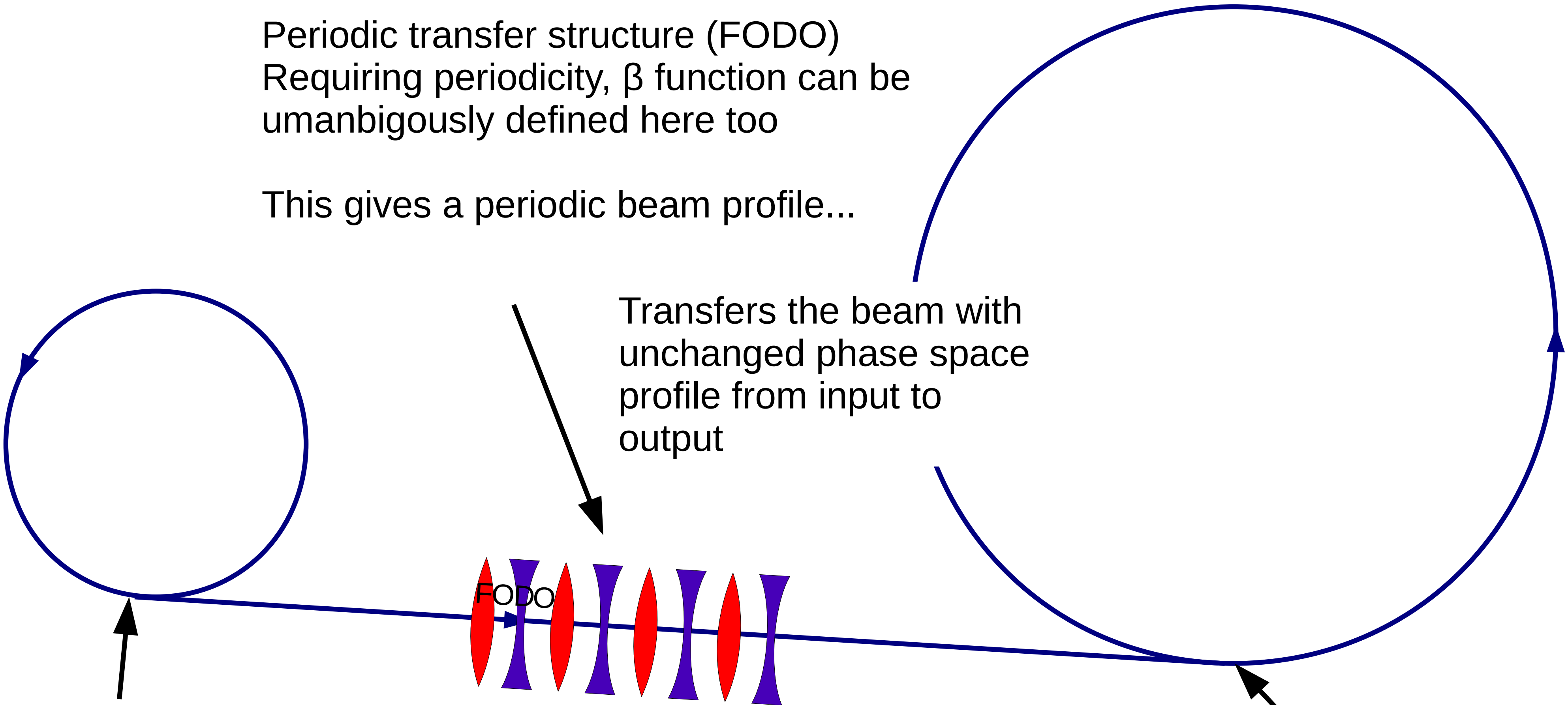
$$\beta\cdot\beta'' - \frac{\beta'^2}{2} + 2k(s)\cdot\beta^2 = 2$$



Twiss parameters in a transfer line

Periodic transfer structure (FODO)
Requiring periodicity, β function can be umanbigously defined here too

This gives a periodic beam profile...



Transfers the beam with unchanged phase space profile from input to output

β function is defined and unique (β_1)

β function is defined and unique (β_2)

Twiss parameters in a transfer line

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… if the beam is prepared correctly (matched) !!!!

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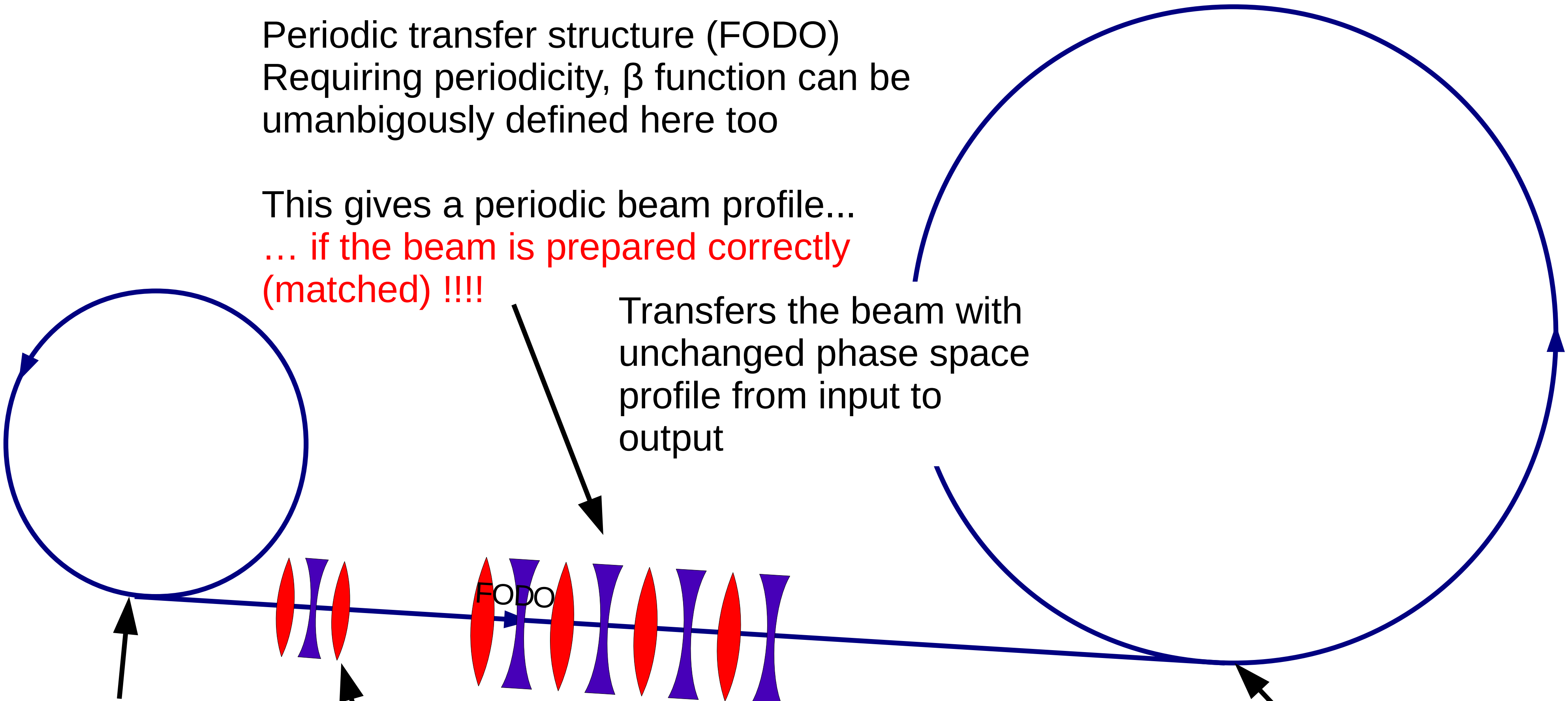

β function is defined and unique (β_1)

Matching section
(few quads, which transform the ellipse of the ring to the eigen-ellipse of the FODO)

β function is defined and unique (β_2)

Twiss parameters in a transfer line

Periodic transfer structure (FODO)
Requiring periodicity, β function can be umanbigously defined here too

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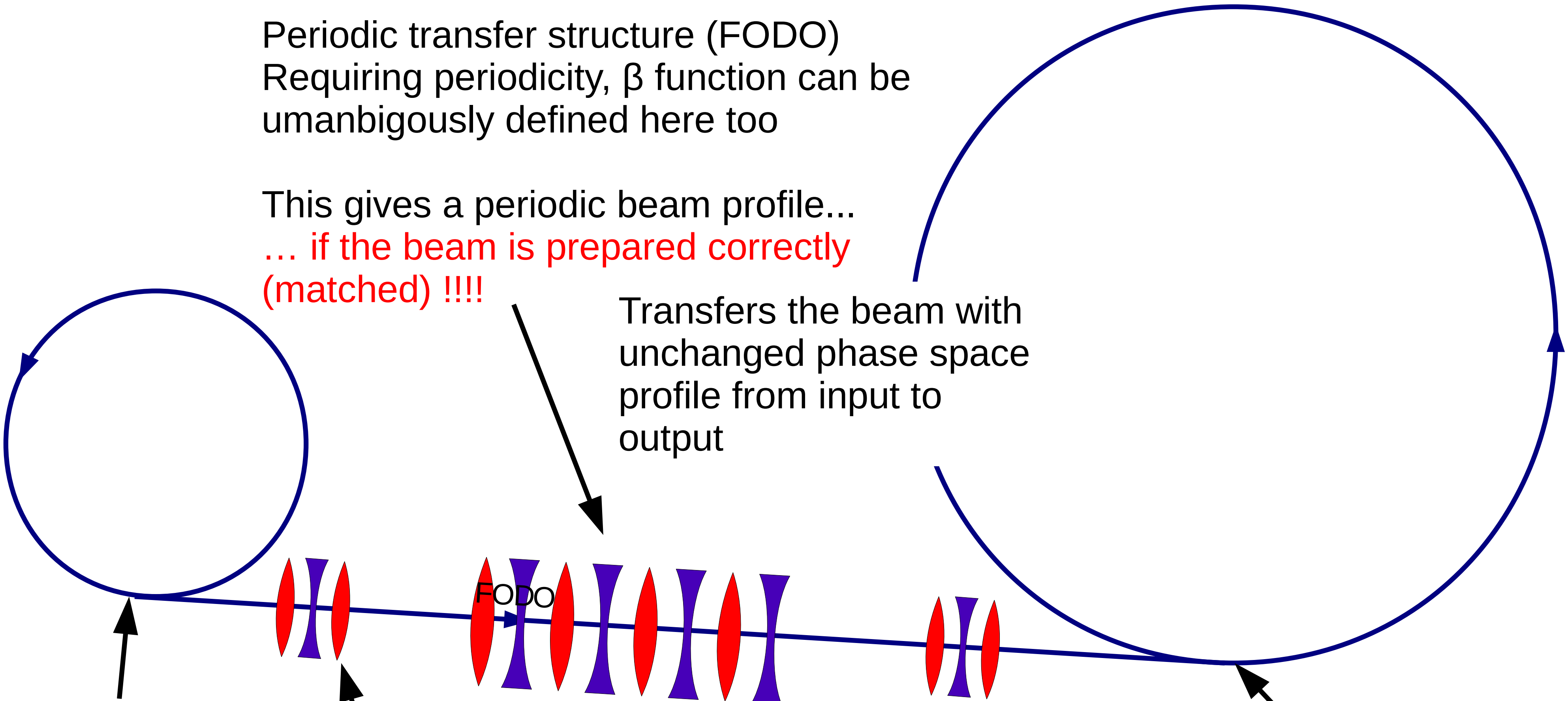

β function is defined and unique (β_1)

Matching section
(few quads, which transform the ellipse of the ring to the eigen-ellipse of the FODO)

Matching section:
FODO → Ring2

β function is defined and unique (β_2)